Math 546, Exam 3, Spring, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points.

The solutions will be posted later today.

(1) (6 points) **Define "group homomorphism". Use complete sentences. Include everything that is necessary, but nothing more.**

If $(G, *)$ and $(G', *')$ are groups, then a group homomorphism from G to G' is a function $\phi : G \to G'$ with $\phi(g_1 * g_2) = \phi(g_1) *' \phi(g_2)$ for all g_1 and g_2 in G.

(2) (6 points) **Define "cyclic group". Use complete sentences. Include everything that is necessary, but nothing more.**

The group $(G, *)$ is cyclic if there is an element g in G with the property that every element in G is equal to $g * g * ... * g$ or $h * h * ... * h$ (where h is the inverse of g) or id.

(3) (6 points) **Define "normal subgroup". Use complete sentences. Include everything that is necessary, but nothing more.**

The subgroup N of the group G is normal if gng^{-1} is in N for all $g \in G$ and $n \in N$.

(4) (6 points) **State and prove the First Isomorphism Theorem.**

Let $\phi : G \to G'$ be a group homomorphism.

(a) If N is a normal subgroup of G with $N \subseteq \ker \phi$, then $\bar{\phi} : \frac{G}{N} \to G'$, given by $\bar{\phi}(gN) = \phi(g)$ is a group homomorphism.

(b) $\bar{\phi} : \frac{G}{\ker \phi} \to \operatorname{im} \phi$ is an isomorphism.

We must verify that $\bar{\phi}$ is a function. (Once we know this, then everything else is obvious.) Suppose that g_1 and g_2 are in G and the cosets $g_1N = g_2N$. We must show that $\phi(g_1) = \phi(g_2)$. The hypothesis $g_1N = g_2N$ ensures that $g_1 = g_2n$ for some $n \in N$. We see that

$$\phi(g_1) = \phi(g_2 n)$$

(5) (6 points) **Consider $\phi : \frac{\mathbb{Z}}{\langle 3\rangle} \to \frac{\mathbb{Z}}{\langle 6\rangle}$, given by $\phi(a + \langle 3\rangle) = a + \langle 6\rangle$. Is ϕ a group homomorphism? Explain.**

NO!! This "ϕ" isn't even a function. The cosets $0 + \langle 3\rangle$ and $3 + \langle 3\rangle$ are equal but "ϕ" sends $0 + \langle 3\rangle$ to $0 + \langle 6\rangle$ and "ϕ" sends $3 + \langle 3\rangle$ to $3 + \langle 6\rangle$. The cosets $0 + \langle 6\rangle$ and $3 + \langle 6\rangle$ are not equal. This makes no sense.

(6) (6 points) **Consider $\phi : \frac{\mathbb{Z}}{\langle 6\rangle} \to \frac{\mathbb{Z}}{\langle 3\rangle}$, given by $\phi(a + \langle 6\rangle) = a + \langle 3\rangle$. Is ϕ a group homomorphism? Explain.**

Yes. Use the first part of the First Isomorphism Theorem. Start with the group homomorphism $\Phi : Z \to \frac{\mathbb{Z}}{\langle 3\rangle}$ with $\Phi(n) = n + \langle 3\rangle$. (This homomorphism is called the natural quotient map. Notice that $\langle 6\rangle$ is contained in the kernel of Φ (which is $\langle 3\rangle$). Hence the First Isomorphism Theorem guarantees that $\bar{\Phi} : \frac{\mathbb{Z}}{\langle 6\rangle} \to \frac{\mathbb{Z}}{\langle 3\rangle}$, with $\bar{\Phi}(n+\langle 6\rangle) = \Phi(n)$ is a group homomorphism. Of course, $\bar{\Phi}$ is the same as ϕ.

(7) (7 points) **Are the groups $\frac{\mathbb{R}}{\mathbb{Z}}$ and $\frac{\mathbb{R}}{2\mathbb{Z}}$ isomorphic? Explain if the answer is no, and give a proof if the answer is yes. (In this problem, $\mathbb{R}$ is the group of real numbers under addition, $\mathbb{Z}$ is the group of integers under addition, and $2\mathbb{Z}$ is the group of even integers under addition.)**

Yes. Consider the composition of two homorphisms

$$\mathbb{R} \xrightarrow{\phi_1} \mathbb{R} \xrightarrow{\phi_2} \frac{\mathbb{R}}{\mathbb{Z}},$$

where $\phi_1(r) = 2r$ and $\phi_2(r) = r + \mathbb{Z}$. Apply the First Isomorphism Theorem to $\phi_2 \circ \phi_1$. Observe that $\phi_2 \circ \phi_1$ is surjective with Kernel equal to $\mathbb{Z}$.

(8) (7 points) **Let N be a normal subgroup of the group G. Suppose a, b, c, d are elements of G and that the cosets aN, bN, cN, and dN satisfy**

$$aN = bN \quad \textbf{and} \quad cN = dN.$$

Do the cosets acN and bdN have to be equal? If yes, prove the statement; if no, explain.

The cosets aN and bN are equal; so there is an element $n_1 \in N$ with $a = bn_1$; and the cosets cN and dN are equal; so there is an element $n_2 \in N$ with $c = dn_2$. Thus,

$$acN = bn_1cN = bc(c^{-1}n_1c)N.$$

The ambient hypothesis ensures that $c^{-1}nc \in N$. It follows that

$$acN = bcN = bdn_2N = bdN.$$