Math 546, Exam 3, Fall, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points. Each problem is worth 10 points.

We use $\mathcal{G}$ to mean the group of rigid motions of the plane under composition and we use D_3 to mean the subgroup of $\mathcal{G}$ generated by σ and ρ, where σ is reflection across the x-axis and ρ is rotation counter-clockwise by $2\pi/3$ fixing the origin. Recall that

$$\sigma \circ \sigma = \text{id}, \quad \rho \circ \rho \circ \rho = \text{id}, \quad \text{and} \quad \rho \circ \sigma = \sigma \circ \rho \circ \rho.$$

The solutions will be posted later today.

(1) **Consider $\phi : D_3 \to D_3$, which is given by $\phi(a) = \sigma \circ a$, for $a \in D_3$. Is ϕ a one-to-one and onto function? If so, prove it. If not, explain why not.**

YES!

We show that ϕ is one-to-one. If a and b are in D_3 with $\phi(a) = \phi(b)$, then $\sigma \circ a = \sigma \circ b$. Multiply both sides on the left by σ's inverse (which is σ) in order to see that $a = b$.

We show that ϕ is onto. If $a \in D_3$, then $\phi(\sigma \circ a) = a$ because $\sigma \circ \sigma = \text{id}$.

(2) **Consider $\phi : D_3 \to D_3$, which is given by $\phi(a) = \sigma \circ a$, for $a \in D_3$. Is ϕ a group homomorphism? If so, prove it. If not, explain why not.**

NO! We proved that every group homomorphism from the group G to the group G' carries the identity element of G to the identity element of G'. However this ϕ carries the identity element of D_3 to σ, which is NOT the identity element of D_3.

(3) **Let S be the set of left cosets of $H = \langle \sigma \rangle$ in D_3. Consider $\Phi : S \times S \to S$, which is given by $\Phi(a \circ H, b \circ H) = (a \circ b) \circ H$, for $a, b \in D_3$. Is Φ a function? If so, prove it. If not, explain why not.**

NO!! A function Φ carries each element of the domain of Φ to one element of the target of Φ. Observe that $(\text{id} \circ H, \rho \circ H)$ and $(\sigma \circ H, \sigma \circ$

$\rho \circ \rho \circ H)$ are equal elements of $S \times S$; however, $\Phi(\mathrm{id} \circ H, \rho \circ H)$ and $\Phi(\sigma \circ H, \sigma \circ \rho \circ \rho \circ H)$ are different elements of S. Indeed,

$$\begin{aligned} \sigma \circ H &= \mathrm{id} \circ H \\ \sigma \circ \rho \circ \rho \circ H &= \rho \circ H, \quad \text{but} \\ \sigma \circ \sigma \circ \rho \circ \rho \circ H &\neq \mathrm{id} \circ \rho \circ H. \end{aligned}$$

The inequality holds because

$$\sigma \circ \sigma \circ \rho \circ \rho \circ H = \rho \circ \rho \circ H = \{\rho \circ \rho, \sigma \circ \rho\}$$

and

$$\rho \circ H = \{\rho, \sigma\}.$$

(4) **Let G the subgroup of $(\mathbb{Z}, +)$ generated by 339, 565, and 791. Which integer generates G? Explain.**

Observe that $\langle 339, 565, 731\rangle =\rangle 113\langle$. Indeed $339 = 3(113)$, $565 = 5(113)$, $791 = 7(113)$ and $113 = 2(339) - 565$.

(5) **Let U_{18} be the group of complex numbers which satisfy the equation $z^{18} = 1$. The operation in U_{18} is multiplication. List the subgroups of U_{18}? Explain.**

There is exactly one subgroup for each divisor d of 18 and that subgroup is $\langle u^{18/d}\rangle$, where $u = e^{2\pi i/18}$. The subgroup $\langle u^{18/d}\rangle$ has order d. The divisors of 18 are $1, 2, 3, 6, 9, 18$. The subgroups of U_{18} are $\langle u^{18}\rangle$, $\langle u^{9}\rangle$, $\langle u^{6}\rangle$, $\langle u^{3}\rangle$, $\langle u^{2}\rangle$, and $\langle u\rangle$.