Math 546, Exam 2 SOLUTIONS, Spring 2010

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **6** problems.

Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

I will post the solutions later today.

1. **(9 points) Let G be a group. Prove that the identity element in G is unique.**

Suppose that e and e' both are identity elements in G. Consider the product ee'. The fact that e is an identity element of G tells us that $ee' = e'$. The fact that e' is an identity element of G tells us that $ee' = e$. We conclude that $e = ee' = e'$.

2. **(7 points) Define *centralizer*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let g be an element in a group G. The *centralizer* of g in G is the set of all elements in G which commute with g. In other words,

$$C(g) = \{x \in G \mid xg = gx\}.$$

3. **(7 points) Define *order*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let g be an element in a group G. The *order* of g is the least positive integer n for which g^n is equal to the identity element of G. If g^n is never equal to the identity element of G, for any positive integer n, then g has infinite order.

4. **(9 points) Let G be an Abelian group and let H be the subset**

$$H = \{g \in G \mid g^2 = \mathbf{id}\}$$

of G. Does H have to be a subgroup of G? If yes, then prove the claim. If no, then give an example.

YES, H is always a subgroup of G as we prove below.

Closure. Take g_1 and g_2 in H. We see that

$$(g_1g_2)(g_1g_2) = (g_1g_1)(g_2g_2) = \text{id}.$$

(The first equality is due to the fact that G is an Abelian group. The second equality is due to the fact that g_1 and g_2 are both in H.) Thus, the product g_1g_2 is also in H.

Inverses. Take $g \in H$. It follows that $g^2 = \text{id}$; thus g is the inverse of g and the inverse of g (which is g itself) is also in H.

Identity. The identity element squares to the identity element; thus, $\text{id} \in H$.

5. **(9 points) Let G be a group and let H be the subset**

$$H = \{g^3 \in G \mid g \in G\}$$

of G. Does H have to be a subgroup of G? If yes, then prove the claim. If no, then give an example.

NO, H is not always a subgroup of G. Consider $G = D_3$. In this case,

$$H = \{\text{id}, \sigma, \sigma\rho, \sigma\rho^2\}$$

and H is not a group because σ and $\sigma\rho$ are in H but the product $\sigma(\sigma\rho) = \rho$ is not in H.

6. **(9 points) List three subgroups of D_4 of order four. (No details are necessary.)**

$$\{\text{id}, \rho, \rho^2, \rho^3\}, \quad \{\text{id}, \sigma, \rho^2, \sigma\rho^2\}, \quad \{\text{id}, \sigma\rho, \rho^2, \sigma\rho^3\}.$$