Math 546, Exam 2, Solutions, Fall 2011

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **8** problems.

Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

I will post the solutions later today.

1. (7 points)**Define** *centralizer*. **Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let g be an element in a group G. The *centralizer* of g in G is the set of all elements in G which commute with g. In other words,

$$C(g) = \{x \in G \mid xg = gx\}.$$

2. (7 points) **Define** *order*. **Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let g be an element in a group G. The *order* of g is the least positive integer n for which g^n is equal to the identity element of G. If g^n is never equal to the identity element of G, for any positive integer n, then g has infinite order.

3. (6 points) **State Lagrange's Theorem.** If H is a subgroup of the finite group G, then the number of elements in H divides the number of elements in G.

4. (6 points) **Prove Lagrange's Theorem.**

The proof has two steps. In step (1) we show that every element of G is in exactly one left coset of H in G. In step (2) we show that every left coset of H in G has the same number of elements as H has. Once we have shown (1) and (2), then we have shown that $|G| = |H| \times |\{aH \mid a \in H\}|$, in other words, the number of elements in G is equal to the number of elements in H times the number of left cosets of H in G.

We prove (1). Take $a \in G$. It is clear that a is in the left coset aH. We show that a is not in any other left coset. Suppose that $a \in bH$ for some b in G. We must show that the sets aH and bH are equal. The fact that $a \in bH$ tells us that $a = bh_0$ for some fixed $h_0 \in H$. Take an arbitrary element ah of aH; so $h \in H$. We see that $ah = bh_0h$. But H is a group and h_0 and h are in H; so $h_0h \in H$ and $ah \in bH$. We have shown that $aH \subseteq bH$. Now, take an arbitrary element bh of bH; so $h \in H$. We have $bh = ah_0^{-1}h$. But H is a group with h_0 and h in H; so $h_0^{-1}h$ is in H and $bh \in aH$. We have shown that $bH \subseteq aH$. We conclude that $aH = bH$.

We prove (2). We establish a one-to-one correspondence between the elements of H and the elements of aH for any fixed left coset aH of H in G. If $h \in H$, then the corresponding element of aH is $\alpha(h) = ah$. If $x \in aH$, then the corresponding element of H is $\beta(x) = a^{\text{inv}}x$. It is clear that $\alpha\colon H \to aH$ and $\beta\colon aH \to H$ are inverses of one another since $\beta(\alpha(h)) = \beta(ah) = a^{\text{inv}}ah = h$ for all $h \in H$, and $\alpha(\beta(x)) = \alpha(a^{\text{inv}}x) = aa^{\text{inv}}x = x$ for all $x \in aH$. It follows that $|H| = |aH|$ for all left cosets aH of H in G.

5. (6 points) **State the result about the relationship between the order of ab, the order of a, and the order of b. Be sure to include all of the hypotheses, but nothing extra.**

Let a and b elements of the group G. Suppose that
(a) a and b have finite order,
(b) the order of a is relatively prime to the order of b, and
(c) $ab = ba$.
Then the order of ab is equal to the order of a times the order of b.

6. (6 points) **Prove the statement in problem 5.**

Let m be the order of a and n be the order of b. It is clear from hypothesis (c) that

$$(ab)^{nm} = (a^m)^n(b^n)^m = (\text{id})^n(\text{id})^m = \text{id}.$$

Thus, the order of ab is at most mn. We must show that the order of ab is at least mn. That is, suppose that r is a positive integer with $(ab)^r = \text{id}$. We must show that $ab \le r$. Well, hypothesis (c) together with the statement $(ab)^r = \text{id}$ tells us that $a^r = (b^{\text{inv}})^r$. Thus, $a^r \in <a> \cap <b>$. The order of a and the order of b are relatively prime; so Lagrange's Theorem tells us that $<a> \cap <b> = \{\text{id}\}$; but $a^r \in <a> \cap <b>$; so $a^r = \text{id}$. It follows that m divides r. Furthermore, $(b^{\text{inv}})^r = a^r = \text{id}$; so, $\text{id} = b^r$. It follows that n divides r. The integers m and n are relatively prime with $m|r$ and $n|r$; hence, $mn|r$. But r is a positive integer; so r is some positive integer multiple of mn. We conclude that $mn \le r$ and the proof is complete.

7. (6 points) **List** 8 **subgroups of** D_4 **in addition to all of** D_4 **and** $\{\mathbf{id}\}$**. A small amount of explanation would be perfect. I am thinking of** D_4 **as the smallest subgroup of** $\mathrm{Sym}(\mathbb{C})$ **which contains** σ **and** ρ**, where** $\mathrm{Sym}(\mathbb{C})$ **is the group of invertible functions from the complex plane to the complex plane (with operation composition),** ρ **is rotation counterclockwise by** $\pi/2$**, and** σ **is reflection across the** x**-axis.**

The non-trivial cyclic subgroups of D_4 are $<\rho> = \{\mathrm{id}, \rho, \rho^2, \rho^3\}$, $<\rho^2> = \{\rho^2, \mathrm{id}\}$, $<\sigma> = \{\sigma, \mathrm{id}\}$, $<\rho\sigma> = \{\rho\sigma, \mathrm{id}\}$, $<\rho^2\sigma> = \{\rho^2\sigma, \mathrm{id}\}$, $<\rho^3\sigma> = \{\rho^3\sigma, \mathrm{id}\}$. In quiz 3, we found that the centralizer of σ in D_4 is $\{\mathrm{id}, \sigma, \rho^2\sigma, \rho^2\}$. The exact same reasoning as we used quiz 3 shows that the centralizer of $\rho\sigma$ in H is $\{\mathrm{id}, \rho\sigma, \rho^3\sigma, \rho^2\}$. We have listed 8 subgroups of D_4.

8. (6 points) **Give an example of a group** G **and elements** a **and** b **in** G **where** a **and** b **each have order** 2**, but** ab **has order** 10**.**

Let $G = \mathrm{Sym}(\mathbb{C})$, a be

$$(\text{rotation by } \tfrac{2\pi}{10}) \circ (\text{reflection across the } x\text{-axis}),$$

and b be reflection across the x-axis. We see that a and b both are reflections; so both of these elements of G have order 2. We also see that ab is rotation by $\frac{2\pi}{10}$, which has oder 10