Math 546, Exam 2, Fall, 2022, Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points. Each problem is worth 10 points.

We use $\mathbb{Q}$ to mean the group of rational numbers under addition and $\mathcal{G}$ to mean the group of rigid motions of the plane under composition.

The solutions will be posted later today.

(1) **Suppose G is a group and every proper subgroup of G is cyclic. Does G have to be cyclic? If so, prove the statement. If not, give an example.**

NO! Let G be the group D_3. The proper subgroups of D_3 all have order 1, 2 or 3 by Lagrange's Theorem. A group with one element is clearly cyclic. Every group of order 2 or 3 is cyclic by Lagrange's Theorem. Of course, D_3 is not cyclic. The rotations have order 3, the reflections have order 2 and every element in D_3 is the identity, a rotation, or a reflection.

(2) **Let H and K be non-zero subgroups of the group $\mathbb{Q}$. Does the intersection of H and K have to be non-zero? If so, prove the statement. If not, give an example.**

YES. The subgroup H has a non-zero element; say $\frac{a}{b}$, where a and b are non-zero integers. In a similar manner, $\frac{c}{d}$ is a non-zero element of K where c and d are non-zero integers. Multiply the numerators and denominators by (-1), if necessary, in order to assume that b and d are both positive integers. Observe that ac, which is equal to both $cb(\frac{a}{b})$ and $ad(\frac{c}{d})$, is a non-zero element of $H \cap K$. (Keep in mind that $cb(\frac{a}{b})$, which is equal to $\frac{a}{b} + \cdots + \frac{a}{b}$, is in H because H is closed under addition.)

(3) **Recall the group D_4, which is the subgroup of $\mathcal{G}$ generated by σ and ρ, where σ is reflection across the x-axis and ρ is rotation by $\pi/2$ radians counterclockwise fixing the origin. Recall that D_4 has eight distinct elements $\sigma^i\rho^j$ with $i \in \{0, 1\}$ and $j \in \{0, 1, 2, 3\}$ and that the generators of D_4 satisfy**

$$\sigma^2 = \mathrm{id}, \quad \rho^4 = \mathrm{id}, \quad \rho\sigma = \sigma\rho^3.$$

Which elements of D_4 commute with $\sigma\rho$? Prove your answer.

Keep in mind that the answer is called the centralizer of $\sigma\rho$ in D_4. You proved for homework that the centralizer of the element a in the group G is a subgroup of G which contains $\langle a \rangle$.

The elements

$$\{\mathrm{id}, \sigma\rho, \rho^2, \sigma\rho^3\}$$

commute with $\sigma\rho$. (To prove this it suffices to check that ρ^2 commutes with $\sigma\rho$ and this is obvious.)

Furthermore, no other elements of D_4 commute with $\sigma\rho$. It is not necessary to check each of the other four elements of D_4. We know that the centralizer of $\sigma\rho$ is a subgroup of D_4 and (by Lagrange's theorem) the only subgroup of D_4 which properly contains the four element set we have found so far is D_4. Consequently, it suffices to prove that any one element of D_4, other than the four we have found so far, does not commute with $\sigma\rho$. In particular

$$\sigma(\sigma\rho) = \rho \quad \text{but} \quad (\sigma\rho)\sigma = \rho^3.$$

Hence σ does not commute with $\sigma\rho$ and the centralizer of $\sigma\rho$ in D_4 is

$$\{\mathrm{id}, \sigma\rho, \rho^2, \sigma\rho^3\}.$$

(4) **Let $(G, *)$ be a group and $H = \{g \in G \mid g * g = \mathrm{id}\}$. Does H have to be a subgroup of G? If so, prove the statement. If not, give an example.**

NO! Take G to be D_3. The three reflections σ, $\sigma\rho$, and $\sigma\rho^2$ have order 2. The identity element has order 1. The two rotations have order 3. The subset H of G is equal to $\{\mathrm{id}, \sigma, \sigma\rho, \sigma\rho^2\}$. This set has four elements. Lagrange's Theorem guarantees that no subgroup of D_3 has four elements. Thus, H is not a subgroup of G.

(5) **Let H be a subgroup of the group $(G, *)$ and let g_1 and g_2 be two elements of G. Suppose that the cosets $g_1 * H$ and $g_2 * H$ have an element in common. Prove that $g_1 * H$ is a subset of $g_2 * H$.**

We are told that $g_1 * h_1 = g_2 * h_2$ for some h_1 and h_2 in G. It follows that $g_1 = g_2 * h_2 * h_1^{-1}$.

Let h be an arbitrary element of H. Observe that

$$g_1 * h = (g_2 * h_2 * h_1^{-1}) * h = g_2 * (h_2 * h_1^{-1} * h),$$

which is an element of $g_2 * H$ because H is a group. We have shown that every element of $g_1 * H$ is in $g_2 * H$.