Math 546, Exam 1, SOLUTIONS Fall 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

1. **Recall that U_{12} is the subgroup $\{1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}\}$ of $(\mathbb{C} \setminus \{0\}, \times)$, with $z = e^{\frac{2\pi \imath}{12}} = \cos(\frac{2\pi}{12}) + \imath \sin(\frac{2\pi}{12})$.**

   (a) **Identify 4 subgroups of U_{12} in addition to $\{1\}$ and U_{12}. Please give a complete explanation.**

   Four subgroups of U_{12} are

   $$\{1, z^2, z^4, z^6, z^8, z^{10}\}, \quad \{1, z^3, z^6, z^9\}, \quad \{1, z^4, z^8\}, \quad \{1, z^6\}.$$

   In each case, the indicated subset is closed under multiplication; and therefore, the subset is a subgroup.

   (b) **Which elements of U_{12} generate U_{12}? (Recall that the element g of the group $(G, *)$ *generates* G if every element of G is equal to $\underbrace{g * g * \dots * g}_{n \text{ times}}$, for some integer n.) Please give a complete explanation.**

   We see that z, z^5, z^7, and z^{11} generate U_{12}. Indeed $z = (z^{11})^{11} \in <z^{11}>$, $z = (z^5)^5 \in <z^5>$, and $z = (z^7)^7 \in <z^7>$. It follows that $<z^{11}>$, $<z^5>$, and $<z^7>$ all are equal to $<z> = U_{12}$. The other elements all generate smaller subgroups of U_{12} as is shown in (a).

2. **Let $S = \mathbb{R} \setminus \{4\}$. Define $*$ on S by $a * b = 20 - 4a - 4b + ab$. Prove that $(S, *)$ is a group.**

Closure: Take a, b from S. We must show that $a * b$ is in S. Well, $a * b = 20 - 4a - 4b + ab$, which is clearly a real number. We must check that $20 - 4a - 4b + ab$ is not equal to 4. If $20 - 4a - 4b + ab$ were equal to 4, then $20 - 4a - 4b + ab = 4$; so, $16 - 4a - 4b + ab = 0$; that is, $(a - 4)(b - 4) = 0$; so $a = 4$ or $b = 4$. On the other hand, a and b are in S; so neither a nor b is 4. We conclude that $20 - 4a - 4b + ab \neq 4$; therefore, $20 - 4a - 4b + ab \in S$

Associativity: Take a, b, and c from S. Observe that

$$a*(b*c) = a*(20-4b-4c+bc) = 20-4a-4(20-4b-4c+bc)+a(20-4b-4c+bc)$$

$$= -60 + 16a + 16b + 16c - 4ab - 4ac - 4bc + abc.$$

On the other hand,

$$(a*b)*c = (20-4a-4b+ab)*c = 20-4(20-4a-4b+ab)-4c+(20-4a-4b+ab)c$$
$$= -60 + 16a + 16b + 16c - 4ab - 4ac - 4bc + abc.$$

We see that $a*(b*c) = (a*b)*c$.

Identity: The number 5 is the identity element of S because

$$a * 5 = 20 - 4a + -4(5) + a(5) = a$$

and $5 * a = 20 - 4(5) - 4a + 5a = a$ for all $a \in S$.

Inverses: Take $a \in S$. The inverse of a is $\frac{15-4a}{4-a}$ because

$$a*\frac{15-4a}{4-a} = 20-4a-4\left(\frac{15-4a}{4-a}\right)+a\left(\frac{15-4a}{4-a}\right) = 20-4a+\frac{(-4+a)(15-4a)}{4-a}$$
$$= 20 - 4a - (15 - 4a) = 5.$$

The operation * is commutative; so, $\frac{15-4a}{4-a} * a$ is also equal to 0. Notice, also, that $\frac{15-4a}{4-a} \in S$ because $\frac{15-4a}{4-a}$ is a real number (since $a \neq 4$) and $\frac{15-4a}{4-a}$ is not equal to 4; because if $\frac{15-4a}{4-a}$ were equal to 4, then $\frac{15-4a}{4-a} = 4$, so $15 - 4a = 4(4-a)$; that is, $15 = 16$, which of course is not possible.

3. **Let G be a group with identity element id. Suppose that H and K are subgroups of G with $H \neq \{\text{id}\}$ and $K \neq \{\text{id}\}$. Is it possible for $H \cap K$ to equal $\{\text{id}\}$? If $H \cap K = \{\text{id}\}$ is possible, then give an example. If $H \cap K = \{\text{id}\}$ is not possible, then give a proof. (Recall that $H \cap K$ is the *intersection* of H and K; that is, $H \cap K = \{g \in G \mid g \in H$ AND $g \in K\}$.)**

Of course, $H \cap K$ is possible. Let G be the Klein 4-group with 4 distinct elements id, a, b, c with identity element id, $a^2 = b^2 = c^2 = \text{id}$, $ba = ab = c$, $ca = ac = b$, $cb = bc = a$. We have seen examples of such groups. Let $H = \{\text{id}, a\}$ and $K = \{\text{id}, b\}$. The sets H and K are closed; hence they are subgroups of G. The intersection of H and K consists only of the identity element id.

4. **Let G be a group. Suppose that H and K are subgroups of G. Does $H \cup K$ have to be a subgroup of G? If $H \cup K$ is always a subgroup of G, then give a proof. If it is possible for $H \cup K$ not to be a subgroup of G, then give an example. (Recall that $H \cup K$ is the *union* of H and K; that is, $H \cup K = \{g \in G \mid g \in H$ OR $g \in K\}$.)**

Of course, $H \cup K$ does NOT have to be a subgroup of G. Take G, H, and K as in problem 3. So H and K are subgroups of G, but $H \cup K$ is not a subgroup of G because $H \cup K$ is not closed because a is in $H \cup K$, b is in $H \cup K$, but $ab = c$ is not in $H \cup K$.

5. **Let** $(G, *)$ **be an Abelian group with identity element** $\mathbf{id}$ **and** $H = \{g \in G \mid g * g * g = \mathbf{id}\}$ **. Prove that** H **is a subgroup of** G **.**

Closure: Take a, b from H so $a * a * a = \mathrm{id}$ and $b * b * b = \mathrm{id}$. The group G is Abelian so

$$(a * b) * (a * b) * (a * b) = (a * a * a) * (b * b * b) = \mathrm{id} * \mathrm{id} = \mathrm{id}.$$

Thus, $a * b$ is in H

Associativity: The operation $*$ is associative on all of G , so $*$ is associative on the subset H of G .

Identity: We are given that id is the identity element of G . It follows that then $\mathrm{id} * \mathrm{id} * \mathrm{id} = \mathrm{id}$ and therefore, $\mathrm{id} \in H$.

Inverses: Let $a \in G$. It follows that $a * a * a = \mathrm{id}$. We are told that G is a group. So a has an inverse, lets call it a^{inv} , in G , Multiply the above equation by a^{inv} to get $\mathrm{id} = a^{\mathrm{inv}} * a^{\mathrm{inv}} * a^{\mathrm{inv}}$. Conclude that a^{inv} is in H .