Math 546, Exam 1, Spring, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points. Each problem is worth 10 points.

The solutions will be posted later today.

(1) **Let H be the subgroup of $(\mathbb{Z},+)$ generated by 2. (Recall that $(\mathbb{Z},+)$ is the group of integers under addition.) What are the elements of H? Explain.**

The even integers are the elements of H. Indeed, H is the subgroup of $(\mathbb{Z},+)$ generated by 2. Every element of H is $2+2+\cdots+2$ or $(-2)+\cdots+(-2)$ or 0.

(2) **Let H be the subgroup of $(\mathrm{GL}_2(\mathbb{R}),\times)$ generated by $A=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$. (Recall that $(\mathrm{GL}_2(\mathbb{R}),\times)$ is the group of 2×2 invertible matrices with real entries under multiplication.) What are the elements of H? Explain.**

The group H has four elements. The elements of H are

$$\left\{A, B, AB=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}, I=\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\}.$$

Observe that $A^2=B^2=(AB)^2=I$ and $AB=BA$.

(3) **Let $(\mathbb{Z},*)$ be the set of integers with operation $a*b=\max\{a,b\}$. (In other words, $a*b$ is equal to the maximum of a and b.) Is $(\mathbb{Z},*)$ a group? Explain.**

The pair $(\mathbb{Z},*)$ is NOT a group. There is no identity element. We prove this assertion by contradiction. If a in $(\mathbb{Z},*)$ were an identity element, then $(a-1)*a$ is equal to $a-1$ because a is the identity element of $(\mathbb{Z},*)$. On the other hand, $(a-1)*a=\max\{a-1,a\}=a$. So

$$a-1=(a-1)*a=a$$

as integers. Add the integer $-a$ to both sides to learn that $-1=0$ as integers. Of course, this is absurd.

(4) **Recall that $\mathcal{G}$ is the group of rigid motions of the xy-plane with operation composition. Let ρ be the element of $\mathcal{G}$ which fixes the orign and rotates the xy-plane counterclockwise by 72 degrees. Let H be the subgroup of $\mathcal{G}$ which is generated by ρ. Write the multiplication table for H.**

The group H has five elements. The elements of H are id, ρ, ρ^2, ρ^3, and ρ^4. The multiplication is

	id	ρ	ρ^2	ρ^3	ρ^4
id	id	ρ	ρ^2	ρ^3	ρ^4
ρ	ρ	ρ^2	ρ^3	ρ^4	id
ρ^2	ρ^2	ρ^3	ρ^4	id	ρ
ρ^3	ρ^3	ρ^4	id	ρ	ρ^2
ρ^4	ρ^4	id	ρ	ρ^2	ρ

(5) (a) **Is it possible for a group to be cyclic, but not Abelian? Explain.**

No. Every cyclic group is Abelian. Indeed, if G is a cyclic group, then G has a generator g and every element of G is g to a power. Every power of g commutes with every power of g.

(b) **Is it possible for a group to be Abelian, but not cyclic? Explain.**

Yes. Every Klein 4 group is Abelian, but not cyclic. So, in particular, the group H of problem 2 is an Abelian group which is not cyclic.