Math 546, Final Exam, Fall, 1994

PRINT Your Name:______________________________

There are 14 problems on 8 pages. The exam is worth a total of 100 points. Problems 8 and 11 are each worth 20 points. The other problems are worth 5 points each.

1. DEFINE *group.*

2. DEFINE *group homomorphism.*

3. What is the order of the element $1-i$ in the group $(\mathbb{C}^*, \times)$? Explain your answer.

4. What is the order of the element $\sqrt{2}/2 - i\sqrt{2}/2$ in the group $(\mathbb{C}^*, \times)$? Explain your answer.

5. Prove that every subgroup of $(\mathbb{Z}, +)$ is cyclic.

6. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.) Let G be a group and let a be a fixed element of G. If $\gamma_a \colon G \to G$, is the function which is given by $\gamma_a(g) = a^{-1}ga$ for all $g \in G$, then γ_a is a permutation of the set G.

7. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.) Let G be a group and let a be a fixed element of G. If $\rho_a \colon G \to G$, is the function which is given by $\rho_a(g) = ga$ for all $g \in G$, then ρ_a is a group homomorphism.

8. All of the following objects are groups. Which of these groups are cyclic groups? Explain each answer.
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_3$
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_3$
   (c) The subgroup $\{e^n \mid n \in \mathbb{Z}\}$ of $(\mathbb{R}^*, \times)$.
   (d) The subgroup $<(1234), (13)(24)>$ of S_4.

9. Record the multiplication table for the group $\dfrac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<(2,2)>}$.

10. Is $\frac{\mathbb{Z}}{3\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}}$, given by $n + 3\mathbb{Z} \mapsto 2n + 9\mathbb{Z}$, a group homomorphism? Explain your answer.

11. In this problem $\mathbb{R}^+$ represents the set of positive real numbers. Which of the following are groups? Explain each answer.
    (a) the set of functions $\{f \colon \mathbb{R}^+ \to \mathbb{R}^+\}$, under composition of functions,
    (b) the set of functions $\{f \colon \mathbb{R}^+ \to \mathbb{R}^+\}$, under multiplication of functions,
    (c) the set of matrices $\{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$, under matrix addition,

12. Give an example of a group G, a subgroup H of G, and elements a, b, and c of G such that $aH = bH$, but $acH \neq bcH$.

13. FILL IN the blank and then PROVE the resulting sentence. If H is a ____________ subgroup of the group G and a, b, and c are elements of G with $aH = bH$, then $acH = bcH$.

14. STATE and PROVE the First Isomorphism Theorem.