Solution to the Quiz for June 3, 2003

Find the general solution of the following system of linear equations:

$$\begin{aligned} x_1 \quad + x_3 + x_4 - 2x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 - x_4 + x_5 &= 0 \\ 3x_1 - x_2 + 4x_3 + x_4 + x_5 &= 1. \end{aligned}$$

CHECK your answer!

The corresponding matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -2 & 1 \\ 2 & 1 & 3 & -1 & 1 & 0 \\ 3 & -1 & 4 & 1 & 1 & 1 \end{array}\right]$$

Replace $R2 \mapsto R2 - 2R1$ and $R3 \mapsto R3 - 3R1$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -3 & 5 & -2 \\ 0 & -1 & 1 & -2 & 7 & -2 \end{array}\right]$$

Replace $R3 \mapsto R3 + R2$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -3 & 5 & -2 \\ 0 & 0 & 2 & -5 & 12 & -4 \end{array}\right]$$

Replace $R3 \mapsto (1/2)R3$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -3 & 5 & -2 \\ 0 & 0 & 1 & -5/2 & 6 & -2 \end{array}\right]$$

Replace $R1 \mapsto R1 - R3$ and $R2 \mapsto R2 - R3$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 7/2 & -8 & 3 \\ 0 & 1 & 0 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & -5/2 & 6 & -2 \end{array}\right]$$

The general sdolution of the original system of equations is:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7/2 \\ 1/2 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}.}$$

We check three specific solutions. When $x_4 = 0$

and $x_5 = 0$, then our solution is

$$\begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

and this is a solution because

$$\begin{aligned} 3 - 2 &= 1 \\ 6 - 6 &= 0 \\ 9 - 8 &= 1. \checkmark \end{aligned}$$

When $x_4 = 2$ and $x_5 = 0$, then our solution is

$$\begin{bmatrix} -4 \\ 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

This really is a solution because

$$\begin{aligned} -4 + 3 + 2 &= 1 \\ -8 + 1 + 9 - 2 &= 0 \\ -12 - 1 + 12 + 2 &= 1. \checkmark \end{aligned}$$

When $x_4 = 0$ and $x_5 = 1$, then our solution is

$$\begin{bmatrix} 11 \\ 1 \\ -8 \\ 0 \\ 1 \end{bmatrix}.$$

This really is a solution because

$$\begin{aligned} &11 - 8 - 2 = 1 \\ &22 + 1 - 24 + 1 = 0 \\ &33 - 1 - 32 + 1 = 1. \checkmark \end{aligned}$$