PRINT Your Name:______________________________

Quiz for June 25, 2007

Let A be an $m \times m$ nonsingular matrix and B be an $m \times n$ matrix. Prove that the column space of AB has the same dimension as the column space of B.

ANSWER: This problem requires some cleverness. We use the fourth theorem about dimension. This Theorem is also known as the rank-nulity theorem. This theorem tells us that the dimension of the column space of AB plus the dimension of the null space of AB is equal to the number of columns of AB. The theorem also tells us that the dimension of the column space of B plus the dimension of the null space of B is equal to the number of columns of B. The matrices AB and B both have n columns. We will prove that the column space of AB has the same dimension as the column space of B by proving that the null space of AB has the same dimension as the null space of B; and we will prove this by showing that the null space of AB **is equal to** the null space of B.

Take a vector x in the null space of B. We see that $ABx = A(0) = 0$ because x is in the null space of B. We conclude that x is in the null space of AB.

Take a vector x in the null space of AB. So, $ABx = 0$. In other words, Bx is a vector that is sent to zero by A. The matrix A is non-singular; so the **only** vector that A sends to 0 is 0. It follows that Bx is already zero, and x is in the null space of B.