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Quiz for October 10, 2006

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.
(c) Express each column of A as a linear combination of the vectors in your answer to (b).

ANSWER: To find a basis for the null space of A, we solve $Ax = 0$. In other words, we apply Elementary Row Operations to A. Apply $R2 \mapsto R2 - 2R1$ and $R3 \mapsto R3 - 2R1$ to get:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

Apply $R1 \mapsto R1 - 2R2$, $R3 \mapsto R3 + 2R2$, and $R4 \mapsto R4 - R2$ to get

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution set of $Ax = 0$ is the set of all vectors x with

$$\begin{array}{rcrr} x_1 & = & x_3 & -2x_4 \\ x_2 & = & -x_3 & +x_4 \\ x_3 & = & x_3 & \\ x_4 & = & & x_4 \end{array}$$

In other words the null space of A is the set of linear combinations of

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

These two vectors are linearly independent (look at rows 3 and 4); so our answer to (a) is

$$\boxed{(a)\quad v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.}$$

The columns in A that correspond to the leading ones in the reduced matrix are a basis for the column space of A. Columns 1 and 2 in the reduced matrix have leading ones; so our basis for the column space of A is columns 1 and 2 of A; in other words, our answer to (b) is

$$\boxed{(b)\quad A_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad A_2 \begin{bmatrix} 2 \\ 5 \\ 2 \\ 1 \end{bmatrix}.}$$

The fact that v_1 is in the null space of A tells us that $A_1 - A_2 + A_3 = 0$; this tells us how to write the third column of A in terms of our answer to (b). The fact that v_2 is in the null space of A tells us that $-2A_1 + A_2 + A_4 = 0$. Our answer to (c) is:

$$\boxed{(c)\quad A_3 = -A_1 + A_2, \quad A_4 = 2A_1 - A_2.}$$