Quiz 1, January 19, 2016

Consider the system of equations

$$\begin{array}{ccccccc} x_1 & +x_2 & & & -x_5 & = & 1 \\ & x_2 & +2x_3 & +x_4 & +3x_5 & = & 1 \\ x_1 & & -x_3 & +x_4 & +x_5 & = & 0 \end{array}$$

(a) Put the corresponding augmented matrix into reduced row echelon form.

(b) What is the general solution of this system of equations?

ANSWER: Start with the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right].$$

Apply $R_3 \mapsto R_3 - R_1$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 - R_2$ and $R_3 \mapsto R_3 + R_2$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + 2R_3$ and $R_2 \mapsto R_2 - 2R_3$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

This matrix is in reduced row echelon from. The solution set is the set of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ such that

$$\begin{aligned} x_1 &= -3x_4 - 6x_5 \\ x_2 &= 1+3x_4 + 7x_5 \\ x_3 &= -2x_4 - 5x_5 \end{aligned}$$

such that x_4 and x_5 are arbitrary.

A different way to say this is to say that the solution set is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 7 \\ -5 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, x_4, x_5 \in \mathbb{R} \right\}}$$

Check. Our answer is correct. When $x_4 = x_5 = 0$ our answer is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} 1 &= 1 \\ 1 &= 1 \\ 0 &= 0. \checkmark \end{aligned}$$

When $x_4 = 1$ and $x_5 = 0$ our answer is

$$\begin{bmatrix} -3 \\ 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} -3 + 4 &= 1 \\ 4 - 4 + 1 &= 1 \\ -3 + 2 + 1 &= 0. \checkmark \end{aligned}$$

When $x_4 = 0$ and $x_5 = 1$ our answer is

$$\begin{bmatrix} -6 \\ 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} -6 + 8 - 1 &= 1 \\ 8 - 10 + 3 &= 1 \end{aligned}$$