Math 544, Exam 1, Solutions Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

There are 5 problems. Each problem is worth 10 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail.**

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations. (Problems 1 and 2 have the same matrix A.)**

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}.$$

2. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}.$$

We do problems 1 and 2 simultaneously We study

$$\left[\begin{array}{ccccc|cc} 1 & 2 & 2 & 1 & 8 & 3 & 3 \\ 1 & 2 & 2 & 2 & 11 & 5 & 5 \\ 2 & 4 & 4 & 3 & 19 & 8 & 9 \end{array}\right]$$

Apply the row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$, to get:

$$\left[\begin{array}{ccccc|cc} 1 & 2 & 2 & 1 & 8 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 2 & 3 \end{array}\right]$$

Apply the row operations $R_1 \mapsto R_1 - R_2$ and $R_3 \mapsto R_3 - R_2$, to get:

$$\left[\begin{array}{ccccc|cc} 1 & 2 & 2 & 0 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{\text{Problem 2 has no solution.}}$$

The general solution Problem 1 is

$$\boxed{\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_3, x_5 \in \mathbb{R} \right\}}$$

Some specific solutions are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

(I took $x_2 = 0$, $x_3 = 0$, and $x_5 = 0$ in v_1; $x_2 = 1$, $x_3 = 0$, and $x_5 = 0$ in v_2; $x_2 = 0$, $x_3 = 1$, and $x_5 = 0$ in v_3; and $x_2 = 0$, $x_3 = 0$, and $x_5 = 1$ in v_4 . These specific solutions work because

$$Av_1 = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+4 \\ 2+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark,$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+2 \\ -1+2+4 \\ -2+4+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark,$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+2 \\ -1+2+4 \\ -2+4+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark,$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 2 & 1 & 8 \\ 1 & 2 & 2 & 2 & 11 \\ 2 & 4 & 4 & 3 & 19 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4-1+8 \\ -4-2+11 \\ -8-3+19 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} . \checkmark$$

3. **Consider the system of linear** equations.

$$(\spadesuit)\qquad \begin{aligned} 4x_1 + ax_2 &= 2 \\ ax_1 + x_2 &= -1. \end{aligned}$$

(a) **Which values for a cause the system to have no solution?**
(b) **Which values for a cause the system to have exactly one solution?**
(c) **Which values for a cause the system to have an infinite number of solutions?**

Explain.

We study

$$\left[\begin{array}{cc|c} 4 & a & 2 \\ a & 1 & -1 \end{array}\right]$$

Apply the row operation $R_2 \mapsto R_2 - \frac{a}{4}R_1$ to get

$$(\diamondsuit)\qquad \left[\begin{array}{cc|c} 4 & a & 2 \\ 0 & 1-\frac{a^2}{4} & -1-\frac{a}{2} \end{array}\right]$$

If $1 - \frac{a^2}{4} \neq 0$, then ($\spadesuit$) a unique solution. Of course, $1 - \frac{a^2}{4} = 0$ exactly when $4 - a^2 = 0$ or $a = 2$ or -2. So far we know that the system of equations ($\spadesuit$) has exactly one solution when $a \neq 2, -2$. If $a = 2$, then ($\diamondsuit$) is

$$\left[\begin{array}{cc|c} 4 & a & 2 \\ 0 & 0 & -2 \end{array}\right].$$

In this case ($\spadesuit$) has no solution. If $a = -2$, then ($\diamondsuit$) is

$$\left[\begin{array}{cc|c} 4 & a & 2 \\ 0 & 0 & 0 \end{array}\right]$$

and ($\spadesuit$) has infinitely many solutions. We conclude that ($\spadesuit$) has

$$\boxed{\begin{cases} \text{exactly one solution} & \text{if } a \neq 2, -2 \\ \text{no solution} & \text{if } a = 2 \\ \text{infinitely many solutions} & \text{if } a = -2 \end{cases}}$$

4. **(True or False. If true, PROVE the result. If false, give a counter EXAMPLE.) If A is a 2×2 matrix with $A^2 = 2A$, then $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.**

This statement is $\boxed{\text{false}}$. The matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ also satisfies $A^2 = 2A$ because

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad 2A = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

5. **(True or False. If true, PROVE the result. If false, give a counter EXAMPLE.) If A and B are 2×2 matrices, then $A^2 - B^2 = (A - B)(A + B)$.**

This statement is $\boxed{\text{false}}$. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Observe that

$$A^2 - B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

On the other hand,

$$(A-B)(A+B) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

thus, $(A - B)(A + B) \neq A^2 - B^2$.