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8. Suppose $v_1, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$. Do $v_1, \ldots, v_n$ have to span $\mathbb{R}^n$? Explain.

Yes. The 1st sentence tells me that $A = [v_1 | \cdots | v_n]$ is an invertible matrix. The IMT tells me that the columns of an invertible $n \times n$ matrix span $\mathbb{R}^n$!

9. Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, which span $\mathbb{R}^m$. Do $v_1, \ldots, v_n$ have to be linearly independent? Explain.

No. There is nothing in the problem which tells me that $n = m$. The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$ but these vectors are not linearly independent because $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

10. Let A, B, and C be 2×2 matrices with A not equal to the zero matrix. Is it possible for $AB = AC$, but $B \neq C$? If possible, find such matrices. If not possible, explain why not.

<u>Yes</u> Here is an example $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

Observe A is not the zero matrix, $B \neq C$,

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

So $AB = AC$.