**Math 544, Final Exam Summer 2007**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

**Please leave room in the upper left corner for the staple.**

There are **9** problems **on TWO sides**. The exam is worth a total of 100 points. Problem 1 is worth 20 points. All of the other problems are worth 10 points each. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website later today.

1. **Let** $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 2 & 3 & 7 \\ 3 & 4 & 6 & 13 \\ 1 & 2 & 6 & 9 \end{bmatrix}$ **. Find a basis for the null space of** $A$ **. Find a basis for the column space of** $A$ **. Find a basis for the row space of** $A$ **. Express each column of** $A$ **as a linear combination of the basis you have chosen for the column space of** $A$ **. Express each row of** $A$ **as a linear combination of the basis you have chosen for the row space of** $A$ **.**

Apply Elementary Row Operations to $A$ to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the null space of $A$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ with

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= -x_4 \\ x_3 &= -x_4 \\ x_4 &= x_4. \end{aligned}$$

We conclude that

$$\boxed{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

is a basis for the nullspace of $A$. The vectors

$$\boxed{A_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, A_{*,2} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, A_{*,3} = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 6 \end{bmatrix}}$$

are a basis for the column space of $A$. The vectors

$$\boxed{\begin{array}{l} w_1 = [1 \quad 0 \quad 0 \quad 1] \\ w_2 = [0 \quad 1 \quad 0 \quad 1] \\ w_3 = [0 \quad 0 \quad 1 \quad 1] \end{array}}$$

are a basis for the row space of $A$. We also see that

$$\boxed{A_{*,4} = A_{*,1} + A_{*,2} + A_{*,3}}$$

and

$$\boxed{\begin{array}{l} A_{1,*} = 1w_1 + 2w_2 + 3w_3 \\ A_{2,*} = 2w_1 + 2w_2 + 3w_3 \\ A_{3,*} = 3w_1 + 4w_2 + 6w_3 \\ A_{4,*} = 1w_1 + 2w_2 + 6w_3 \end{array}}$$

2. **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in the vector space $V$ are linearly independent if the only numbers $c_1, \dots, c_p$ with $c_1v_1 + \cdots + c_pv_p = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

3. **Let $T\colon V \to W$ be a linear transformation of vector spaces. Let $w_1, \dots, w_b$ be a basis for the image of $T$; $z_1, \dots, z_a$ be a basis for the null space of $T$; and $v_1, \dots, v_b$ be vectors in $V$ with $T(v_i) = w_i$ for $1 \le i \le b$. Prove that the vectors $v_1, \dots, v_b, z_1, \dots, z_a$ span $V$. Recall that the image of $T$ is equal to**

$$\{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

Let $v$ be a vector in $V$. The vector $T(v)$ is in the image of $T$; so, $T(v)$ is a linear combination of the vectors $w_1, \dots, w_b$ (which form a basis for the image of $T$). In other words, there are numbers $c_1, \dots, c_b$ with $T(v) = \sum_{i=1}^{b} c_i w_i$. The hypothesis tells us that $w_i = T(v_i)$ for all $i$; hence

$$T(v) = \sum_{i=1}^{b} c_i T(v_i).$$

Move everything to one side and use the fact that $T$ is a linear transformation to see that

$$T(v - \sum_{i=1}^{b} c_i v_i) = 0.$$

In other words, $v - \sum_{i=1}^{b} c_i v_i$ is in the null space of $T$. The vectors $z_1, \dots, z_a$ form a basis for the null space of the null space of $T$; hence, $v - \sum_{i=1}^{b} c_i v_i$ can be written as a linear combination of $z_1, \dots, z_a$ and $v$ can be written as a linear combination of $z_1, \dots, z_a, v_1, \dots, v_b$.

4. **Let $U \subseteq V \subseteq W$ be vector spaces. Suppose that $v_1, v_2, v_3, v_4$ is a basis for $W$. Suppose further that $v_1, v_2, v_3$ are in $V$, but $v_4$ is not in $V$. Suppose finally, that $v_1$ and $v_2$ are in $U$, but $v_3$ and $v_4$ are not in $U$. What is the dimension of $U$? Explain your answer VERY THOROUGHLY.**

The dimension of $U$ is $\boxed{2}$. The vector space $V$ is a proper suspace of the four dimensional vector space $W$ (so $\dim V \le 3$); furthermore $V$ contains 3 linearly independent vectors; hence, $\dim V = 3$. The vector space $U$ is a proper suspace of the three dimensional vector space $V$ (so $\dim U \le 2$); furthermore $U$ contains 2 linearly independent vectors; hence, $\dim U = 2$.

5. **Let $V = \{p(x) \in \mathcal{P}_3 \mid p'(1) = 0\}$. Is $V$ a vector space? If yes, then find a basis for $V$. If no, then show why not? (Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree less than or equal to $3$.)**

YES. The polynomials

$$\boxed{1, \quad (x-1)^2, \quad (x-1)^3}$$

form a basis for $V$.

6. **Let $V = \{M \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \mathrm{tr}(M) = 0\}$. Is $V$ a vector space? If yes, then find a basis for $V$. If no, then show why not? (Recall that $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ is the vector space of $3 \times 3$ matrices. The trace of the $3 \times 3$ matrix**

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix}$$

**is $\mathrm{tr}(M) = m_{1,1} + m_{2,2} + m_{3,3}$.)**

YES. The matrices

$$\boxed{\begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix}}$$

form a basis for $V$.

7. **Let**

$$v = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \quad u_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad u_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \quad u_3 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \quad \textbf{and} \quad u_4 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}.$$

**Express $v$ as a linear combination of $u_1$, $u_2$, $u_3$, $u_4$. (You are encouraged to notice that $u_1, u_2, u_3, u_4$ form an orthogonal set of vectors.)**

Write $v = \sum_{i=1}^{4} c_i u_i$. Multiply by $u_1^{\mathrm{T}}$ to see that $10 = 4c_1$ (i.e., $c_1 = \frac{5}{2}$). Multiply by $u_2^{\mathrm{T}}$ to see that $c_2 = -\frac{1}{2}$. Continue in this manner to see $c_3 = c_4 = -1$. We conclude that $\boxed{\frac{5}{2}u_1 - \frac{1}{2}u_2 - u_3 - u_4 = v.}$

8. **Find a matrix $B$ with $B^2$ equal to $A = \begin{bmatrix} -11 & 15 \\ -20 & 24 \end{bmatrix}$.**

Notice that $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of $A$ which belongs to $4$ and $\begin{bmatrix}3\\4\end{bmatrix}$ is an eigenvector of $A$ which belongs to $9$. Thus, $AS = SD$ for $S = \begin{bmatrix}1&3\\1&4\end{bmatrix}$ and $D = \begin{bmatrix}4&0\\0&9\end{bmatrix}$. We take

$$B = S\begin{bmatrix}2&0\\0&3\end{bmatrix}S^{-1} = \begin{bmatrix}2&9\\2&12\end{bmatrix}\begin{bmatrix}4&-3\\-1&1\end{bmatrix} = \boxed{\begin{bmatrix}-1&3\\-4&6\end{bmatrix}}$$

(Do be sure to check your answer. Mine works.)

9. **Find an orthogonal basis for the null space of** $A = [1 \quad 2 \quad 1 \quad 2]$**.**

One basis for the null space of $A$ is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 5u_2' = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}.$$

(At this point be sure to check that $u_2$ is in the null space of $A$ and that $u_2$ is perpendicular to $u_1$.) Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} -5 \\ -10 \\ -5 \\ 15 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}.$$

Let $u_3 = -3u_3' = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$. Be sure to verify that $u_3$ is in the null space of $A$ and that $u_3$ is orthogonal to $u_1$ and $u_2$. Our basis for the null space of $A$ is:

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}.}$$