Math 544, Exam 3, Summer 2007

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are **6** problems **on TWO sides**. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) **Let** $V = \{p(x) \in \mathcal{P}_3 \mid \int\limits_0^1 p(x)dx = 0\}$ **. Is** V **a vector space? If yes, then find a basis for** V **. If no, then show why not? (Recall that** $\mathcal{P}_3$ **is the vector space of polynomials of degree less than or equal to** 3 **.)**

Yes. The polynomials

$$x - \tfrac{1}{2}, \quad x^2 - \tfrac{1}{3}, \quad x^3 - \tfrac{1}{4}$$

are a basis for V .

2. (7 points) **Let** V **be the set of singular** 2×2 **matrices. Is** V **a vector space? If yes, then find a basis for** V **. If no, then show why not?**

NO. The set V is not closed under addition. Indeed $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in V , but $M_1 + M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not in V .

3. (7 points) **Let** $U \subseteq V$ **be vector spaces. Suppose that** v_1, v_2, v_3, v_4 **is a basis for** V **. Suppose further that** v_1 **and** v_2 **are in** U **, but** v_3 **and** v_4 **are not in** U **. What is the dimension of** U **? Explain your answer VERY THOROUGHLY.**

The dimension of U might be 2 or the dimension of U might be 3 . Those are the only two choices and both could happen. First of all, v_1, v_2 is the beginning of a basis for U ; hence, $2 \le \dim U$. On the other hand, if U some how had a basis which contained 4 or more vectors, then these 4 linearly independent vectors would necessarily be a basis for V ; so, in this case U would equal V . But U doesn't equal V ; hence, U 's bases all have 3 or fewer vectors.

Here are two examples to show that $\dim U$ could equal 2 or 3 :

- If v_1 and v_2 are a basis for U , then dim $U = 2$.
- Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

and let U be the span of v_1 , v_2 , and u_3 . It is clear that v_1, v_2, u_3 are linearly independent, hence $\dim U = 3$. It is also clear that neither v_3 nor v_4 is in U (otherwise both v_3 and v_4 would have to be in U and U would be equal to all of V .)

4. (7 points) **Let A be a 3×5 matrix. Suppose that z_1 , z_2 , x_1 , and x_2 are vectors in $\mathbb{R}^5$ and y_1 and y_2 are vectors in $\mathbb{R}^3$. Suppose further that $Az_1 = 0$, $Az_2 = 0$, $Ax_1 = y_1$, and $Ax_2 = y_2$. Suppose finally, that z_1 and z_2 are linearly independent, and that y_1 and y_2 are linearly independent. Do z_1, z_2, x_1, x_2 have to be linearly independent? If yes, give a complete proof. If no, give a counter example.**

Yes. Suppose c_1 , c_2 , c_3 , and c_4 are numbers with

$$(*) \qquad c_1 z_1 + c_2 z_2 + c_3 x_1 + c_4 x_2 = 0.$$

Multiply by A to get

$$c_3 y_1 + c_4 y_2 = 0.$$

The vectors y_1 and y_2 are linearly independent; hence c_3 and c_4 must be zero. So the original equation (*) is

$$c_1 z_1 + c_2 z_2 = 0.$$

However, the vectors z_1 and z_2 are linearly independent; hence, c_1 and c_2 must also be zero. The only numbers which cause (*) to happen are $c_1 = c_2 = c_3 = c_4 = 0$; and therefore, z_1, z_2, x_1, x_2 are linearly independent.

5. (11 points) **In this problem, if M is a matrix, then let $\mathcal{I}(M)$ denote the column space of M. Let A and B be $n\times n$ matrices. Answer each question. If the answer is yes, then give a proof. If the answer is no, then give a counter example.**

(a) **Is $\mathcal{I}(A)$ always a subset of $\mathcal{I}(AB)$?**

No. If $A=I$ and $B=0$, then $\mathcal{I}(A)=\mathbb{R}^n$ but $\mathcal{I}(AB)=\{0\}$ and $\mathbb{R}^n$ is not a subset of $\{0\}$.

(b) **Is $\mathcal{I}(B)$ always a subset of $\mathcal{I}(AB)$?**

No. If $B=I$ and $A=0$, then $\mathcal{I}(B)=\mathbb{R}^n$ but $\mathcal{I}(AB)=\{0\}$ and $\mathbb{R}^n$ is not a subset of $\{0\}$.

(c) **Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(A)$?**

Yes. If $v\in\mathcal{I}(AB)$, then $v=ABx$ for some vector $x\in\mathbb{R}^n$. It follows that v is also equal to A times the vector Bx; and therefore $v\in\mathcal{I}(A)$

(d) **Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(B)$?**

No. Take $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&0\end{bmatrix}$. Observe that $AB=\begin{bmatrix}0&0\\1&0\end{bmatrix}$. We see that $\begin{bmatrix}0\\1\end{bmatrix}$ is in the column space of AB, but $\begin{bmatrix}0\\1\end{bmatrix}$ is not in the column space of B.

(e) **Suppose B is non-singular. Is $\mathcal{I}(A)$ always a subset of $\mathcal{I}(AB)$?**

Yes. If $v\in\mathcal{I}(A)$, then $v=Ax$ for some $x\in\mathbb{R}^n$; hence, $v=AB(B^{-1}x)$ which is in the column space of AB.

(f) **Suppose B is non-singular. Is $\mathcal{I}(B)$ always a subset of $\mathcal{I}(AB)$?**

No. If $B=I$ and $A=0$, then $\mathcal{I}(B)=\mathbb{R}^n$ but $\mathcal{I}(AB)=\{0\}$ and $\mathbb{R}^n$ is not a subset of $\{0\}$.

(g) **Suppose B is non-singular. Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(A)$?**

YES. This case is covered in (c).

(h) **Suppose B is non-singular. Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(B)$?**

YES. If B is non-singular, then $\mathcal{I}(B)=\mathbb{R}^n$ and it certainly is true that $\mathcal{I}(AB)$ is a subset of $\mathbb{R}^n$.

6. (11 points) **Let** $A=\begin{bmatrix}1&7&8&5&1&8&7&2&11\\1&7&8&5&2&14&11&2&11\\1&7&8&5&2&14&11&3&13\\3&21&24&15&5&36&29&7&35\end{bmatrix}$. **Find a basis for the null space of A. Find a basis for the column space of A. Find a basis for the row space of A. Express each column of A as a linear**

combination of the basis you have chosen for the column space of A. Express each row of A as a linear combination of the basis you have chosen for the row space of A. Apply $R2 \mapsto R2 - R1$, $R3 \mapsto R3 - R1$, $R4 \mapsto R4 - 3R1$ to obtain

$$\begin{bmatrix} 1 & 7 & 8 & 5 & 1 & 8 & 7 & 2 & 11 \\ 0 & 0 & 0 & 0 & 1 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 12 & 8 & 1 & 2 \end{bmatrix}.$$

Apply $R1 \mapsto R1 - R2$, $R3 \mapsto R3 - R2$, $R4 \mapsto R4 - 2R2$ to obtain

$$\begin{bmatrix} 1 & 7 & 8 & 5 & 0 & 2 & 3 & 2 & 11 \\ 0 & 0 & 0 & 0 & 1 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Apply $R1 \mapsto R1 - 2R3$, $R4 \mapsto R4 - R3$ to obtain

$$\begin{bmatrix} 1 & 7 & 8 & 5 & 0 & 2 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The null space of A is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}$ with

$$\begin{array}{rcccccc}
x_1 = & -7x_2 & -8x_3 & -5x_4 & -2x_6 & -3x_7 & -7x_9 \\
x_2 = & x_2 & & & & & \\
x_3 = & & x_3 & & & & \\
x_4 = & & & x_4 & & & \\
x_5 = & & & & -6x_6 & -4x_7 & \\
x_6 = & & & & x_6 & & \\
x_7 = & & & & & x_7 & \\
x_8 = & & & & & & -2x_9 \\
x_9 = & & & & & & x_9
\end{array}$$

It follows that the vectors

$$\boxed{v_1 = \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -8 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_6 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

is a basis for the null space of A. (As always, I write $A_{*,j}$ for column j of the matrix A and $A_{i,*}$ for row i of the matrix A.) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \ A_{*,5} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix}, \ A_{*,8} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 7 \end{bmatrix}}$$

are a basis for the column space of A. The vectors

$$\boxed{\begin{aligned} w_1 &= [1 \quad 7 \quad 8 \quad 5 \quad 0 \quad 2 \quad 3 \quad 0 \quad 7], \\ w_2 &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 6 \quad 4 \quad 0 \quad 0], \\ w_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2] \end{aligned}}$$

are a basis for the row space of A. We see that

$$\boxed{\begin{array}{ccc} A_{*,2} = 7A_{*,1}, & A_{*,3} = 8A_{*,1}, & A_{*,4} = 5A_{*,1}, \\ A_{*,6} = 2A_{*,1} + 6A_{*,5}, & A_{*,7} = 3A_{*,1} + 4A_{*,4}, & A_{*,9} = 7A_{*,1} + 2A_{*,8}. \end{array}}$$

We also see that

$$\boxed{\begin{aligned} A_{1,*} &= w_1 + w_2 + 2w_3 \\ A_{2,*} &= w_1 + 2w_2 + 2w_3 \\ A_{3,*} &= w_1 + 2w_2 + 3w_3 \\ A_{4,*} &= 3w_1 + 5w_2 + 7w_3 \end{aligned}}$$