Math 544, Exam 2, Summer 2007, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are **9** problems **on TWO sides**. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. cc

You should **KEEP** this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (6 points) **Let A be a fixed $n\times n$ matrix and let $W=\{x\in\mathbb{R}^n \mid Ax=2x\}$. Is W a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.**

YES! We see that W is the null space of the matrix $A-2I_n$. The null space of every matrix is a vector space.

2. (6 points) **Let $W=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2 \,\middle|\, |x_1|=|x_2|\right\}$. Is W a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.**

NO! The vectors $w_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $w_2=\begin{bmatrix}1\\-1\end{bmatrix}$ are in W, but $w_1+w_2=\begin{bmatrix}2\\0\end{bmatrix}$ is not in W.

3. (6 points) **Define "null space". Use complete sentences. Include everything that is necessary, but nothing more.**

The <u>null space</u> of the matrix A is the set of all column vectors x with $Ax=0$.

4. (6 points) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The $n\times n$ matrix A is <u>non-singular</u> if the only vector x in $\mathbb{R}^n$ with $Ax=0$ is $x=0$.

5. (6 points) **Let A be an $n \times n$ matrix. List three statements that are equivalent to the statement "A is non-singular". Do not repeat your answer to problem 4.**

1. The columns of A are linearly independent.
2. The system of equations $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
3. The matrix A is invertible.

6. (5 points) **Let A and B be symmetric $n \times n$ matrices. Does the matrix AB HAVE to be symmetric? If yes, PROVE the statement. If no, give an EXAMPLE.**

NO! The matrices $A = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are both symmetric; but the matrix $AB = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is not symmetric.

7. (5 points) **Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Suppose that v_1, v_2, v_3, v_4 are linearly independent. Do the vectors v_1, v_2, v_3 HAVE to be linearly independent? If yes, PROVE the result. If no, show an EXAMPLE.**

YES! Suppose that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. It follows that

(*)
$$c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0.$$

The vectors v_1, v_2, v_3, v_4 are linearly independent; hence the only coefficients which cause (*) to happen are $c_1 = c_2 = c_3 = 0$. We conclude that v_1, v_2, v_3 are linearly independent.

8. (5 points) **Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}}v_j = 0$ for all subscripts i and j with $i \neq j$. Prove that v_1, v_2, and v_3 are linearly independent.**

Suppose c_1, c_2, and c_3 are numbers with

(*)
$$c_1v_1 + c_2v_2 + c_3v_3 = 0.$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}}v_1 + c_2 \cdot v_1^{\mathrm{T}}v_2 + c_3 \cdot v_1^{\mathrm{T}}v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}}v_2 = 0$ and $v_1^{\mathrm{T}}v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}}v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}}v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (*) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}}v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}}v_2 \neq 0$. Multiply (*) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.

9. (5 points) **Consider the vectors**

$$w = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}?$$

Is the vector w in the span of the vectors v_1, v_2, and v_3? Explain thoroughly.

The question asks if $Ax = w$ has a solution where

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

We apply the technique of Gaussian Elimination to

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 2 & 5 & 8 & 8 \\ 3 & 6 & 9 & 10 \end{array}\right]$$

Replace $R2 \mapsto R2 - 2R1$ and $R3 \mapsto R3 - 3R1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 0 & -3 & -6 & -6 \\ 0 & -6 & -12 & -11 \end{array}\right]$$

Replace $R3 \mapsto R3 - 2R2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 0 & -3 & -6 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The bottom row shows that $Ax = w$ does not have a solution. It follows that w is NOT in the span of the vectors v_1, v_2, and v_3.