Math 544, Exam 1, Summer 2007 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 7 problems. The exam is worth a total of 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum\limits_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

2. (7 points) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is *non-singular* if the only column vector x with $Ax = 0$ is $x = 0$.

3. (7 points) **Let A be an $n \times n$ matrix. List two conditions which are equivalent to the statement "A is non-singular". Do not repeat your answer to problem 2.**

The following conditions are equivalent to the statement "A is non-singular".
(1) The columns of A are linearly independent.
(2) The system of equations $Ax = b$ has a unique solution for each vector $b \in \mathbb{R}^n$.

4. (8 points) **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{rrrrrcl} x_1 & +3x_2 & +4x_3 & +2x_4 & +4x_5 & = & 16 \\ x_1 & +3x_2 & +4x_3 & +3x_4 & +6x_5 & = & 21 \\ 2x_1 & +6x_2 & +8x_3 & +5x_4 & +10x_5 & = & 37 \end{array}$$

Consider

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 2 & 4 & 16 \\ 1 & 3 & 4 & 3 & 6 & 21 \\ 2 & 6 & 8 & 5 & 10 & 37 \end{array}\right]$$

Apply $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 2 & 4 & 16 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Apply $R_3 \mapsto R_3 - R_2$ and $R_1 \mapsto R_1 - 2R_2$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution of this system of equations is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.}$$

Some specific solutions are:

$$\begin{bmatrix} 6 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

(We took $x_2 = x_3 = x_5 = 0$, $x_2 = 1$ with $x_3 = x_5 = 0$, $x_3 = 1$ with $x_2 = x_5 = 0$, $x_5 = 1$ with $x_2 = x_3 = 0$.) We check that each specific solution does indeed satisfy the equations:

$$\begin{array}{rrr} 6 + 10 = 16 & 3 + 3 + 10 = 16 & 2 + 4 + 10 = 16 \\ 6 + 15 = 21 & 3 + 3 + 15 = 21 & 2 + 4 + 15 = 21 \\ 12 + 25 = 37 & 6 + 6 + 25 = 37 & 4 + 8 + 25 = 37 \end{array}$$

$$\begin{array}{r} 6 + 6 + 4 = 16 \\ 6 + 9 + 6 = 21 \\ 12 + 15 + 10 = 37. \checkmark \end{array}$$

5. (7 points) **Consider the system of linear equations.**

$$\begin{aligned} x_1 + 4a x_2 &= 4 \\ a x_1 + x_2 &= 2. \end{aligned}$$

(a) **Which values for a cause the system to have no solution?**
(b) **Which values for a cause the system to have exactly one solution?**
(c) **Which values for a cause the system to have an infinite number of solutions?**

Explain thoroughly.

Apply row operations to

$$\left[\begin{array}{cc|c} 1 & 4a & 4 \\ a & 1 & 2 \end{array}\right]$$

Apply $R_2 \mapsto R_2 - aR_1$ to get

$$\left[\begin{array}{cc|c} 1 & 4a & 4 \\ 0 & 1-4a^2 & 2-4a \end{array}\right]$$

If $1-4a^2 \neq 0$, then the system of equations has a unique solution.
If $1-4a^2 = 0$ and $2-4a = 0$, then the system of equations has infinitely many solutions.
If $1-4a^2 = 0$ and $2-4a \neq 0$, then the system of equations has no solution.
We notice that $1-4a^2 = 0$ when $(1-2a)(1+2a) = 0$. Thus, $a = \frac{1}{2}$, or $a = -\frac{1}{2}$.
We notice that when $a = \frac{1}{2}$, then $2-4a = 0$
We notice that when $a = -\frac{1}{2}$, then $2-4a \neq 0$.
We conclude

If a is different than $\frac{1}{2}$ and $-\frac{1}{2}$, then the system has a unique solution. If $a = \frac{1}{2}$, then the system has infinitely many solutions. If $a = -\frac{1}{2}$, then the system has no solution.

6. (7 points) **Are the vectors**

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

linearly independent? Explain thoroughly.

These vectors are linearly dependent because $v_1 - 2v_2 + v_3 = 0$.

7. (7 points) **Let** v_1, v_2, v_3, v_4 **be vectors in** $\mathbb{R}^5$**. Suppose that** v_1, v_2, v_3, v_4 **are linearly dependent. Do the vectors** v_1, v_2, v_3 **HAVE to be linearly dependent? If yes, PROVE the result. If no, show an EXAMPLE.**

No. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly dependent, but $v_1 . v_2, v_3$ are linearly independent.