Math 544, Final Exam, Summer 2006, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 12 problems. The exam is worth a total of 100 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

I will post the solutions on my website shortly after class is finished.

1. (7 points) **Define "span". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, v_2, \dots, v_n$ in the vector space V *span* V if every vector in V is equal to a linear combination of $v_1, v_2, \dots, v_n$.

2. (7 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in the vector space V are *linearly independent* if the ONLY numbers $c_1, \dots, c_p$, with $c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$ are $c_1 = c_2 = \dots = c_p = 0$.

3. (10 points) **Consider the matrices**

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}.$$

(a) **Find the GENERAL solution of the system of linear equations $Ax = b$.**

Apply the row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to the matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 10 & 5 \\ 1 & 2 & -1 & 2 & 16 & 5 \\ 2 & 4 & -2 & 3 & 26 & 10 \end{array}\right]$$

to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 10 & 5 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{array}\right].$$

Apply $R_3 \mapsto R_3 - R_2$ and $R_1 \mapsto R_1 - R_2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution of the system of equations is

$$\boxed{\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix} \,\middle|\, x_2, x_3, x_5 \in \mathbb{R} \right\}.}$$

(b) **List three SPECIFIC solutions of $Ax = b$.**

Some specific solutions of this system of equations are

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix}.$$

(In v_1, I took $x_2 = x_3 = x_5 = 0$. In v_2, I took $x_2 = 1$, $x_3 = 0$, and $x_5 = 0$. In v_3, I took $x_2 = 0$, $x_3 = 1$, and $x_5 = 0$. In v_4, I took $x_2 = 0$, $x_3 = 0$, and $x_5 = 1$.)

(c) **CHECK that the specific solutions satisfy the equations.**

We check

$$Av_1 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 \\ 5 \cdot 1 \\ 5 \cdot 2 \end{bmatrix} = b.\checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 2 + 1 \cdot 4 \end{bmatrix} = b.\checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 2 + 1 \cdot (-2) \end{bmatrix} = b.\checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 6 \cdot 1 + 1 \cdot 10 \\ 1 \cdot 1 - 6 \cdot 2 + 1 \cdot 16 \\ 1 \cdot 2 - 6 \cdot 3 + 1 \cdot 26 \end{bmatrix} = b.\checkmark$$

(d) **Find a basis for the null space of A.**

The vectors

$$\boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(e) **Find a basis for the column space of A.**

The vectors

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

are a basis for the column space of A.

(f) **Find a basis for the row space of A.**

The vectors

$$\boxed{w_1 = [\,1 \quad 2 \quad -1 \quad 0 \quad 4\,] \quad \text{and} \quad w_2 = [\,0 \quad 0 \quad 0 \quad 1 \quad 6\,]}$$

are a basis for the row space of A.

(g) **Express each column of A in terms of your answer to (e).**

We see that

$$\boxed{A_{*,1} = u_1, \quad A_{*,2} = 2u_1, \quad A_{*,3} = -u_1, \quad A_{*,4} = u_2, \quad A_{*,5} = 4u_1 + 6u_2,}$$

where $A_{*,j}$ means column j of the matrix A.

(h) **Express each row of A in terms of your answer to (f).**

We see that

$$\boxed{A_{1,*} = w_1 + w_2, \quad A_{2,*} = w_1 + 2w_2, \quad A_{3,*} = 2w_1 + 3w_2,}$$

where $A_{i,*}$ means row i of the matrix A.

4. (8 points) **Find an orthogonal basis for the null space of $A = [\,1 \quad 3 \quad 4 \quad 5\,]$.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$u_1 = v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{12}{10} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \left(\begin{bmatrix} -20 \\ 0 \\ 5 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 5u_2' = \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

We check that $u_1^{\mathrm{T}} u_2 = 0$ and $Au_2 = 0$. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{15}{10} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{10}{65} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{13} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{26} \left(\begin{bmatrix} -130 \\ 0 \\ 0 \\ 26 \end{bmatrix} - 39 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{26} \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.$$

Let

$$u_3 = 26u_3' = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.$$

Check that $Au_3 = 0$, $u_1^{\mathrm{T}} u_3 = 0$, and $u_2^{\mathrm{T}} u_3 = 0$. Our answer is

$$\boxed{u_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.}$$

5. (8 points) **Let** $A = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix}$**. Find a matrix** B **with** $B^2 = A$**. CHECK your answer.**

Find the eigenvalues and eigenvectors of A.

$$0 = \det(A - \lambda I) = (-1-\lambda)(14-\lambda) + 50 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9).$$

So the eigenvalues of A are 4 and 9. The eigenspace belonging to 4 is spanned by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The eigenspace belonging to 9 is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let

$$S = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

Observe that $AS = SD$. We see that $S^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$. Let

$$B = S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}}.$$

We check that

$$B^2 = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix} = A.$$

6. (10 points) **State the four Theorems about vector space dimension.**

Theorem 1. If V is a vector space with one finite basis, then every basis for V has the same number of vectors.

Theorem 2. If V is a (finite dimensional) vector space, then every linearly independent subset in V is part of a basis for V.

Theorem 3. If V is a vector space, then every finite spanning set for V contains a basis for V.

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A.

7. (10 points) **Let A be an $n \times n$ matrix. Record eight statements that are equivalent to "the matrix A is invertible".**

1. There is a matrix B with AB equal to the identity matrix and BA equal to the identity matrix.
2. There is a matrix B with AB equal to the identity matrix.
3. There is a matrix B with BA equal to the identity matrix.
4. The null space of A is $\{0\}$.
5. The columns of A are linearly independent.
6. The only solution to $Ax = 0$ is $x = 0$.
7. The columns of A span $\mathbb{R}^n$.
8. The system of equations $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.
9. The columns of A are a basis for $\mathbb{R}^n$.
10. The dimension of the null space of A is zero.
11. The dimension of the column space of A is n.
12. The rank of A is n.
13. The rows of A are linearly independent.
14. The rows of A span the vector space of all row vectors with n entries.
15. The dimension of the row space of A is n.
16. Zero is not an eigenvalue of A.
17. The determinant of A is not zero.
18. The matrix A is non-singular.

8. (7 points) **Let V be a subspace of $\mathbb{R}^4$. Suppose v_1, v_2, v_3 are linearly independent elements of V. Suppose also that $V \neq \mathbb{R}^4$. Does v_1, v_2, v_3 have to be a basis for V? If yes, prove your assertion. If no, give a counter example.**

YES. The vector space V is a proper subspace of $\mathbb{R}^4$; so $\dim V < \dim \mathbb{R}^4 = 4$. Thus, $\dim V \leq 3$. The linearly independent set v_1, v_2, v_3 is contained in a basis for V; but every basis for V has $\dim V$ vectors and $\dim V \leq 3$. Thus, v_1, v_2, v_3 MUST already be a basis for V.

9. (8 points) **Let V be the vector space of symmetric 3×3 matrices. Give a basis for V. Explain your answer.**

One basis for V is

$$\boxed{\begin{matrix} \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, & \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, & \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \\[2ex] \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, & \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, & \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}. \end{matrix}}$$

It is clear that each of the listed matrices is symmetric. It is clear that every 3×3 symmetric matrix is a linear combination of the six listed matrices. It is also clear that the six listed matrices are linearly independent.

10. (8 points) **Let V be the set of non-singular 2×2 matrices. Is V a vector space? Explain your answer.**

NO. The set V is not closed under addition. The matrices $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $B = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ are both in V; but the sum $A+B = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not in V.

11. (7 points) **Let $\mathcal{C}$ be the vector space of integrable functions $f\colon \mathbb{R} \to \mathbb{R}$, and let $F\colon \mathcal{C} \to \mathbb{R}$ be the function which is defined by $F(f) = \int_1^2 f(x)dx$ for each $f \in \mathcal{C}$. Is F a linear transformation? Explain your answer.**

YES. Use the rules of calculus:

$$F(f+g) = \int_1^2 [f(x)+g(x)]dx = \int_1^2 f(x)dx + \int_1^2 g(x)dx = F(f) + F(g)$$

and

$$F(cf) = \int_1^2 cf(x)dx = c\int_1^2 f(x)dx = cF(f)$$

for all $f, g \in \mathcal{C}$ and all $c \in \mathbb{R}$.

12. (10 points) **In this problem, if M is a matrix, then let $\mathcal{N}(M)$ be the null space of M. Let A and B be $n \times n$ matrices. For each question: if the answer is yes, then prove the statement; if the answer is no, then give a counter example.**

   (a) **Does $\mathcal{N}(B)$ have to be a subset of $\mathcal{N}(AB)$?**
   (c) **Suppose B is non-singular. Does $\mathcal{N}(B)$ have to be a subset of $\mathcal{N}(AB)$?**
   (e) **Suppose A is non-singular. Does $\mathcal{N}(B)$ have to be a subset of $\mathcal{N}(AB)$?**

Parts (a), (c), and (e) all have answer YES. If $v \in \mathcal{N}(B)$, then $Bv = 0$; so, $ABv = 0$ and $v \in \mathcal{N}(AB)$. The singularity or non-singularity of A and/or B is completely irrelevant.

   (b) **Does $\mathcal{N}(AB)$ have to be a subset of $\mathcal{N}(B)$?**

   (d) **Suppose B is non-singular. Does $\mathcal{N}(AB)$ have to be a subset of $\mathcal{N}(B)$?**

Parts (b) and (d) both have answer NO. If A is the zero matrix and B is the identity matrix, then $\mathcal{N}(AB) = \mathbb{R}^n$, $\mathcal{N}(B) = \{0\}$ and $\mathbb{R}^n$ is not a subset of $\{0\}$. Of course, in this example, B is non-singular.

   (f) **Suppose A is non-singular. Does $\mathcal{N}(AB)$ have to be a subset of $\mathcal{N}(B)$?**

YES. If $v \in \mathcal{N}(AB)$, then $ABv = 0$. However, this matrix A is non-singular and $A(Bv) = 0$; so Bv must already be zero. Thus, v must already be in $\mathcal{N}(B)$.