Math 544, Exam 1, Summer 2006 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points. There are 7 problems. Problem one is worth 8 points. Each of the other problems is worth 7 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after class is finished.

1. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}.$$

We apply row operations to

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 1 & 4 & 5 & 2 & 10 & 5 \\ 3 & 12 & 15 & 4 & 26 & 11 \end{array}\right].$$

Replace $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 - 3R1$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right].$$

Replace $R1 \mapsto R1 - R2$ and $R3 \mapsto R3 - R2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution of the system of equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_2, x_3, and x_5 are free to take any value.

To obtain particular solutions of the system of equations, take $x_2 = x_3 = x_5 = 1$ to obtain v_1 ; $x_2 = 1$, $x_3 = x_5 = 0$ to obtain v_2 ; $x_2 = x_5 = 0$, $x_3 = 1$ to obtain v_3 and $x_2 = x_3 = 0$, $x_5 = 1$ to obtain v_4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We check

$$Av_1 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

and

$$Av_4 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b. \ \checkmark$$

2. **Consider the system of linear equations.**

$$\begin{aligned} x_1 + a x_2 &= 1 \\ a x_1 + 4 x_2 &= 2. \end{aligned}$$

- **(a) Which values for a cause the system to have no solution?**
- **(b) Which values for a cause the system to have exactly one solution?**
- **(c) Which values for a cause the system to have an infinite number of solutions?**

Explain thoroughly.

Apply row operations to

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ a & 4 & 2 \end{array}\right].$$

Replace $R2 \mapsto R2 - aR1$ to obtain

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 4 - a^2 & 2 - a \end{array}\right].$$

We conclude:
(a) If $4 - a^2 = 0$ and $2 - a$ is not zero, then the system of equations has no solution.

That is, the system of equations has no solution for $a = -2$.

(b) If $4 - a^2 \neq 0$, then the system of equations has exactly one solution.

The system of equations has exactly one solution whenever a is different from both 2 and -2.

(c) If $4 - a^2 = 0$ and $2 - a = 0$, then the system of equations has an infinite number of solutions.

That is, the system of equations has an infinite number of solutions when $a = 2$.

3. **Let A and B be 2×2 matrices with AB invertible. Does A have to be invertible? If yes, prove your answer. If no, give a counterexample.**

YES. We are told that there is a 2×2 matrix C with $C(AB) = I$ and $(AB)C = I$. Thus, BC is a a 2×2 matrix with $A(BC) = I$. We proved in class (twice) that if M and N are $n \times n$ matrices with $MN = I$, then NM is also equal to I. Thus, $(BC)A$ is also equal to I and BC is the inverse of A.

4. **Recall that the matrix A is** *symmetric* **if $A^{\mathrm{T}} = A$. Let A and B be 2×2 symmetric matrices with $AB = BA$. Does AB have to be symmetric? If yes, prove your answer. If no, give a counterexample.**

YES. We proved that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ for all matrices A and B. The hypothesis that A and B are symmetric ensures that $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$. Now use the hypothsis that $AB = BA$ to see that:

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = BA = AB.$$

We conclude that AB is a symmetric matrix.

5. **Let v_1, v_2, v_3 be linearly independent vectors in $\mathbb{R}^3$. Can every vector in $\mathbb{R}^3$ be written in terms of v_1, v_2, v_3 in a unique way? If yes, prove your answer. If no, give a counterexample.**

YES. Let M be the matrix whose columns are v_1, v_2, v_3. The matrix M is a 3×3 matrix and the columns of M are linearly independent; thus, the Non-Singular Matrix Theorem applies to M. Thus M satisfies ALL of the conditions of the Non-Singular Matrix Theorem. In particular, the system of equations $Mx = v$ has a unique solution x for all column vectors $v \in \mathbb{R}^3$. In other words, once the vector $v \in \mathbb{R}^3$ is given there, then there is a unique choice of scalars x_1, x_2, and x_3 with $x_1 v_1 + x_2 v_2 + x_3 v_3 = v$.

6. **Let A and B be 2×2 matrices with A not equal to the zero matrix and $BA = A^2$. Does B have to equal A? If yes, prove your answer. If no, give a counterexample.**

NO. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 6 \\ 0 & 7 \end{bmatrix}$. We see that A is not the zero matrix and BA and A^2 are both equal to the zero matrix, but $B \neq A$.

7. **Let v_1, v_2, v_3 be vectors in $\mathbb{R}^3$. Suppose that v_1 and v_2 are linearly independent; v_1 and v_3 are linearly independent; and v_2 and v_3 are linearly independent. Do v_1, v_2, v_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

NO. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. We see that v_1 and v_2 are linearly independent; v_1 and v_3 are linearly independent; v_2 and v_3 are linearly independent; and v_1, v_2, v_3 are not linearly independent because $v_1 + v_2 - v_3 = 0$.