Math 544, Final Exam , Summer 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. Problem 1 is worth 20 points. Each of the other problems is worth 8 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will e-mail your grade to you.

I will post the solutions on my website shortly after the class is finished.

1. **Let**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \textbf{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

(a) **Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.**
(b) **Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.**
(c) **Find a basis for the null space of A.**
(d) **Find a basis for the column space of A.**
(e) **Find a basis for the row space of A.**
(f) **Express each column of A in terms of your answer to (d).**
(g) **Express each row of A in terms of your answer to (e).**

We study the augmented matrix

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 4 & 6 & 2 & 1 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 2 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 1 & 4 & 2 & 3 \end{array}\right].$$

Apply $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 2R_1$, and $R_4 \mapsto R_4 - 2R_1$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Exchange rows 2 and 3 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_2$ and $R_4 \mapsto R_4 - R_2$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_3$ and $R_4 \mapsto R_4 - R_3$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiply rows 2 and 3 by -1 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The general solution to $Ax = b$ is

$$\boxed{\textbf{(a)} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{for any } x_2,\ x_3,\ x_6 \text{ in } \mathbb{R}.}$$

Four specific solutions are

$$\boxed{v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}.}$$

(I obtained v_1 by setting $x_2 = x_3 = x_6 = 0$; v_2 by setting $x_2 = 1, x_3 = x_6 = 0$; v_3 by setting $x_3 = 1, x_2 = x_6 = 0$; and v_4 by setting $x_6 = 1, x_2 = x_3 = 0$.) I check that

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b. \checkmark$$

(b) The equations $Ax = c$ have $\boxed{\text{NO solution.}}$

(c) The vectors

$$\boxed{w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(d) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix}1\\2\\2\\2\end{bmatrix}, \quad A_{*,4} = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \quad A_{*,5} = \begin{bmatrix}1\\1\\2\\1\end{bmatrix}}$$

are a basis for the column space of A.

(e) The vectors

$$\boxed{\begin{aligned} z_1 &= [1 \quad 2 \quad 3 \quad 0 \quad 0 \quad 1] \\ z_2 &= [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1] \\ z_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] \end{aligned}}$$

are a basis for the row space of A.

(f)

$$\boxed{A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}.}$$

(g)

$$\boxed{\begin{aligned} A_{1,*} &= z_1 + z_2 + z_3, \\ A_{2,*} &= 2z_1 + 2z_2 + z_3, \\ A_{3,*} &= 2z_1 + z_2 + 2z_3, \\ A_{4,*} &= 2z_1 + z_2 + z_3. \end{aligned}}$$

2. **Let $U \subseteq V$ be vector spaces. Is it always true that $\dim U \leq \dim V$? If yes, prove your answer. If no, give an example.**

YES. Every basis for U is a linearly independent set in U; hence every basis for U is a linearly independent set V. One of the dimension theorems says that every linearly independent subset of a vector space V may be extended to become a basis for V. Thus, $\dim U \leq \dim V$.

3. **Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. Suppose v_1, v_2, and v_3 are linearly independent in V. Do $T(v_1)$, $T(v_2)$, and $T(v_3)$ have to be linearly independent in W? If yes, prove your answer. If no, give an example.**

NO. Let $V = W = \mathbb{R}^3$, T be the linear transformation which sends every vector to zero, and v_1, v_2, and v_3 be the standard basis for V. We see that v_1, v_2, and v_3 are linearly independent, but $T(v_1)$, $T(v_2)$, and $T(v_3)$ are linearly dependent.

4. **Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. Suppose v_1, v_2, and v_3 are vectors in V and $T(v_1)$, $T(v_2)$, and $T(v_3)$ are linearly independent in W. Do v_1, v_2, and v_3 have to be linearly independent in V? If yes, prove your answer. If no, give an example.**

YES. Suppose c_1, c_2, and c_3 are numbers with $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in V. Apply the linear transformation T to see that

$$T(c_1v_1 + c_2v_2 + c_3v_3) = T(0).$$

The function T is a linear transformation so the left side is equal to $c_1T(v_1) + c_2T(v_2) + c_3T(v_3)$ and the right side is 0. The vectors $T(v_1)$, $T(v_2)$, and $T(v_3)$ are linearly independent; thus, the ONLY linear combination of these vectors which adds to zero has $c_1 = c_2 = c_3 = 0$.

5. **Let A be an $n \times n$ matrix. Let v_1 and v_2 be non-zero vectors in $\mathbb{R}^n$ with $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where λ_1 and λ_2 are distinct real numbers. Prove that v_1 and v_2 are linearly independent.**

Suppose

$$c_1v_1 + c_2v_2 = 0. \tag{1}$$

Multiply both sides of (1) by A to get

$$c_1\lambda_1v_1 + c_2\lambda_2v_2 = 0. \tag{2}$$

Multiply both sides of equation (1) by λ_2 to get

$$c_1\lambda_2v_1 + c_2\lambda_2v_2 = 0. \tag{3}$$

Subtract (2) minus (3) to get

$$c_1(\lambda_1 - \lambda_2)v_1 = 0.$$

The vector v_1 is not zero. If a scalar times v_1 is zero, then the scalar must be zero. Thus, the scalar $c_1(\lambda_1 - \lambda_2) = 0$. But, $(\lambda_1 - \lambda_2)$ is not zero; so, c_1 must be zero. Equation (1) now says that $c_2v_2 = 0$. The vector v_2 is not zero; so, the scalar c_2 must be zero.

6. **Let** $A = \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & 2 & -2 \\ 2 & 0 & 0 & 3 \end{bmatrix}$ **. Find the inverse of** A **. You may do the problem any way you like; however, you might want to notice that the columns of** A **form an orthogonal set.**

We see that

$$A^{\mathrm{T}}A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}.$$

Multiply both sides of the equation on the left by

$$\begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{21} \end{bmatrix},$$

to see that

$$\begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{21} \end{bmatrix} A^{\mathrm{T}}A = I.$$

Thus, the inverse of A is

$$\begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{21} \end{bmatrix} A^{\mathrm{T}} = \boxed{\begin{bmatrix} \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 \\ -\frac{2}{21} & -\frac{2}{21} & -\frac{2}{21} & \frac{1}{7} \end{bmatrix}}$$

Check that $AA^{-1} = A^{-1}A = I$.

7. **Let** $A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ -\frac{3}{2} & \frac{7}{4} \end{bmatrix}$ **. Find** $\lim\limits_{n\to\infty} A^n$ **.**

Diagonalize A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

So,

$$\lim_{n\to\infty} A^n = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \lim_{n\to\infty} \begin{bmatrix} \left(\frac{1}{4}\right)^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}}.$$

8. **Find a basis for the vector space spanned by**

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}.$$

Find a basis for the column space of

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 2 & 4 & 6 & 7 \\ 3 & 6 & 7 & 8 \\ 4 & 8 & 8 & 9 \end{bmatrix}$$

Apply

$$R_2 \mapsto R_2 - 2R_1, \quad R_3 \mapsto R_3 - 3R_1, \quad R_4 \mapsto R_4 - 4R_1$$

to get

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & -8 & -10 \\ 0 & 0 & -12 & -15 \end{bmatrix}.$$

Apply

$$R_3 \mapsto R_3 - 2R_2, \quad R_4 \mapsto R_4 - 3R_2$$

to get

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Apply

$$R_2 \mapsto -\frac{1}{4} R_2$$

to get

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 - 5R_2$ to get

$$\begin{bmatrix} 1 & 2 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\boxed{v_1, v_3}$ is a basis for the vector space spanned by v_1, v_2, v_3, v_4. It is clear that v_1 and v_3 are linearly independent. Furthermore, $v_2 = 2v_1$, and $v_4 = \frac{1}{4}(5v_3 - v_1)$.

9. **The *trace* of the square matrix M is the sum of the elements on the main diagonal of M. Let V be the vector space of all 3×3 matrices M with the trace of M equal to zero. Find a basis for V.**

The matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are all in V. These matrices are clearly linearly independent. So, $\dim V \geq 8$. On the other hand, V is a proper subspace of $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ but is not in V. It follows that $\dim V < \dim \mathrm{Mat}_{3\times 3}(\mathbb{R}) = 9$. Thus, $\dim V = 8$ and the eight matrices that we have listed form a basis for V.

10. **Recall that $\mathcal{P}_4$ is the vector space of all polynomials of degree less than or equal to four. Let W be the subspace of all polynomials in $\mathcal{P}_4$ which satisfy $p(1) + p(-1) = 0$ and $p(2) + p(-2) = 0$. What is the dimension of W?**

Consider the linear transformation $T\colon \mathcal{P}_4 \to \mathbb{R}^2$, which is given by $T(p(x)) = \begin{bmatrix} p(1) + p(-1) \\ p(2) + p(-2) \end{bmatrix}$. The vector space W is the null space of T. So the dimension of W is equal to the dimension of $\mathcal{P}_4$ minus the dimension of the image of T. We know that $\dim \mathcal{P}_4 = 5$, since $1, x, x^2, x^3, x^4$ is a basis for $\mathcal{P}_4$. The image of T is all of $\mathbb{R}^2$ because the image of T is a subspace of $\mathbb{R}^2$ which contains $T(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. The vectors $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ span $\mathbb{R}^2$. We conclude that

$$\dim W = 5 - 2 = \boxed{3}.$$

There are many other ways to reach this answer. The most straightforward thing to do is to calculate a basis for W. One such basis is $x, x^3, (x^2 - 1)(x^2 - 4)$.

11. **Let** $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **. Find vectors** u_2 **,** u_3 **, and** u_4 **in** $\mathbb{R}^4$ **so that** u_1, u_2, u_3, u_4 **is an orthogonal set.**

You could apply the Gram-Schmidt procedure to

$$u_1, v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

I choose v_2, v_3, v_4 to be perpendiclar to u_1 . Or you could apply the Gram-Schmidt procedure to

$$u_1, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Of course, Gram-Schmidt will end up telling us that one of these 5 vectors can be written interms of the other four. But that is fine. At any rate, one orthogonal basis that contains u_1 is

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}, u_4 = \begin{bmatrix} -2 \\ -4 \\ -6 \\ -7 \end{bmatrix}.}$$

Be sure to notice that my vectors do form an orthogonal set.