Math 544, Exam 4, Summer 2005 Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 7 problems. Problem 1 is worth 8 points. Each of the other problems is worth 7 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Let $U \subseteq V$ be subspaces of $\mathbb{R}^n$ with $\dim U = \dim V$. Do U and V HAVE to be equal? If yes, prove your answer. If no, give an example.**

YES. Let $u_1, \dots, u_r$ be a basis for U. Thus $u_1, \dots, u_r$ is a linearly independent set in the vector space V. One of the dimension theorems tells us that $u_1, \dots, u_r$ is the begining of a basis for V; that is, we may adjoin more vectors to this list, if necessary, to get a basis for V. However every basis for V has $\dim V$ vectors and $\dim V = \dim U = r$. Thus, $u_1, \dots, u_r$ is already a basis for V. Thus U and V are both spanned by $u_1, \dots, u_r$ and $U = V$.

2. **Define "span". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$, in the vector space V, *span* V if every vector in V is equal to a liner combination of the vectors $v_1, \dots, v_p$.

3. **Define "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.**

Let V and W be vector spaces. The function $T\colon V \to W$ is a linear transformation if $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv_1) = cT(v_1)$ for all vectors v_1 and v_2 in V and all scalars c in $\mathbb{R}$.

4. **Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with**

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\5\\8\end{bmatrix} \quad \textbf{and} \quad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}3\\-2\\1\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$.

We see that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\4\end{bmatrix} - 2\begin{bmatrix}1\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) - 2T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\-2\\1\end{bmatrix} - 2\begin{bmatrix}2\\5\\8\end{bmatrix}$$
$$= \begin{bmatrix}-1\\-12\\-15\end{bmatrix}.$$

5. **Find an orthogonal basis for the null space of $A = [1 \quad 1 \quad 1 \quad 2]$. CHECK your answer.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}.$$

Let $u_1 = v_1$. Let

$$w_2 = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix} - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\\0\end{bmatrix}.$$

Let

$$u_2 = 2w_2 = \begin{bmatrix}-1\\-1\\2\\0\end{bmatrix}.$$

Before you go any further, be sure to notice that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$. Let

$$w_3 = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix}-2\\0\\0\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} - \frac{2}{6}\begin{bmatrix}-1\\-1\\2\\0\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{2}{3}\\1\end{bmatrix}.$$

Let

$$u_3 = 3w_3 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$

be sure to check $Au_3 = 0$, $u_1^{\mathrm{T}} u_3 = 0$, and $u_2^{\mathrm{T}} u_3 = 0$. Our answer is

$$\boxed{u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}}.$$

6. **Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \sqrt{3}\,x$. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$. CHECK your answer.**

The line $y = \sqrt{3}\,x$ makes the angle $\theta = \frac{\pi}{3}$ with the x-axis. (If need be draw the right triangle with base 1 and height $\sqrt{3}$. The hypothenus is 2. So the angle of inclination, θ, has $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$ and $\sin\theta = \frac{\text{op}}{\text{hyp}} = \frac{\sqrt{3}}{2}$. Thus $\theta = \frac{\pi}{3}$.) It follows that

$$M = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}.$$

The best check is to make sure that $Mv = v$ for some vector on $y = \sqrt{3}\,x$ (like for example $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$); and $Mw = -w$ for some vector perpendicular to $y = \sqrt{3}\,x$ (like for example $w = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$). This happens.

7. **Let $A = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix}$. Find a matrix B with $B^2 = A$. CHECK your answer.**

Find the eigenvalues and eigenvectors of A.

$$0 = \det(A - \lambda I) = (-1-\lambda)(14-\lambda) + 50 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9).$$

So the eigenvalues of A are 4 and 9. The eigenspace belonging to 4 is spanned by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The eigenspace belonging to 9 is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let

$$S = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

Observe that $AS = SD$. We see that $S^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$. Let

$$B = S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}}.$$

We check that

$$B^2 = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix} = A.$$