**Math 544, Exam 2, Summer 2005 Solutions**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 7 problems. Problem 1 is worth 8 points. Each of the other problems is worth 7 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Consider the system of linear equations.**

$$\begin{aligned} x_1 + \quad\quad ax_2 &= 1 \\ ax_1 + (3a-2)x_2 &= 2. \end{aligned}$$

(a) **Find all values of $a$ which cause the system to have no solution?**
(b) **Find all values of $a$ which cause the system to have exactly one solution?**
(c) **Find all values of $a$ which cause the system to have an infinite number of solutions?**

**Explain thoroughly.**

Apply the row operation $R_2 \mapsto R_2 - aR_1$ to the matrix

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ a & 3a-2 & 2 \end{array}\right]$$

to obtain

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & -a^2+3a-2 & 2-a \end{array}\right]$$

If $-a^2+3a-2 \neq 0$, then the system of equations has a unique solution. Of course, $-a^2+3a-2=0$, when $a^2-3a+2=0$, that is, $(a-2)(a-1)=0$.

$\boxed{\text{(b) If } a \neq 1, 2, \text{ then the system of equations has a unique solution.}}$

If $a = 2$ , then the system of equations is:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 2. \end{aligned}$$

These two equations represent the same line. There are infinitely many points on this line.

(c) If $a = 2$, then the system of equations has infinitely many solutions.

If $a = 1$ , then the system of equations is

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2. \end{aligned}$$

These equations represent parallel lines. Parallel lines do not intersect.

(a) If $a = 1$, then the system of equations has no solution.

2. **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^m$ are *linearly independent* if the ONLY numbers $c_1, \dots, c_p$ , with $c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$ are $c_1 = c_2 = \dots = c_p = 0$ .

3. **Define "linear combination". Use complete sentences. Include everything that is necessary, but nothing more.**

Let $v_1, \dots, v_p$ , and $v$ be vectors in $\mathbb{R}^m$ . The vector $v$ is a *linear combination* of the vectors $v_1, \dots, v_p$ if there exist numbers $c_1, \dots, c_p$ , with
$c_1v_1 + c_2v_2 + \dots + c_pv_p = v$ .

4. **Let $A$ be an $n \times n$ matrix. List three statements that are equivalent to "$A$ is non-singular".**

The following statements are equivalent.

(0) The matrix $A$ is non-singular. (That is, the only vector $x$ with $Ax = 0$ is the zero vector.)
(1) The columns of $A$ are linearly independent.
(2) The system of equations $Ax = b$ has a unique solution for all column vectors $b$ in $\mathbb{R}^n$ .
(3) The matrix $A$ is invertible.

5. **Let $A$ and $B$ be $n \times n$ matrices with $AB$ equal to the identity matrix. PROVE $BA$ is equal to the identity matrix. ("We did this in class" is not a satisfactory answer. I expect a complete, coherent proof. You are allowed to use any relevant part of problem 4.)**

We first see that the matrix $B$ is non-singular. Indeed, if $x$ is a column vector with $Bx = 0$, then $ABx = A0$; so, $Ix = 0$; that is, $x = 0$. We have established that the only vector $x$ with $Bx = 0$ is $x = 0$. This tells us that $B$ is non-singular.

Apply problem (4) to conclude that $B$ has an inverse. This inverse is a matrix $C$ with $BC = CB = I$.

Our proof is complete once we show that $C = A$. Look at the product $ABC$. On the one hand, $ABC = (AB)C = IC = C$. On the other hand, $ABC = A(BC) = AI = A$. Thus, $A = C$, and $BA = BC = I$.

6. **Find the general solution of the following system of linear equations.**

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + \phantom{2}x_4 &= 2 \\ x_1 + 2x_2 + 4x_3 + 2x_4 &= 3. \end{aligned}$$

**Also find three particular solutions of this system of equations. Be sure to check that all three of your particular solutions really satisfy the original system of linear equations.**

Apply $R_2 \mapsto R_2 - R_1$ to

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 4 & 2 & 3 \end{array}\right]$$

to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 - 3R_2$ to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

The general solution of the system of equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

where $x_2$ and $x_4$ are free to take any value.

Three particular solutions are

$$v_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For $v_1$ , I took $x_2 = x_4 = 0$ . For $v_2$ , I took $x_2 = 1$ and $x_4 = 0$ . For $v_3$ , I took $x_2 = 0$ and $x_4 = 1$ . I checked that each particular solution works.

7. **Let $v_1$ , $v_2$ , and $v_3$ be non-zero vectors in $\mathbb{R}^4$ . Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts $i$ and $j$ with $i \neq j$ . Prove that $v_1$ , $v_2$ , and $v_3$ are linearly independent.**

Suppose $c_1$ , $c_2$ , and $c_3$ are numbers with

(*) $$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

Multiply by $v_1^{\mathrm{T}}$ to get

$$c_1 \cdot v_1^{\mathrm{T}} v_1 + c_2 \cdot v_1^{\mathrm{T}} v_2 + c_3 \cdot v_1^{\mathrm{T}} v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}} v_2 = 0$ and $v_1^{\mathrm{T}} v_3 = 0$ . So, $c_1 \cdot v_1^{\mathrm{T}} v_1 = 0$ . The hypothesis also tells us that $v_1$ is not zero; from which it follows that $v_1^{\mathrm{T}} v_1 \neq 0$ . We conclude that $c_1 = 0$ . Multiply (*) by $v_2^{\mathrm{T}}$ to see that $c_2 \cdot v_2^{\mathrm{T}} v_2 = 0$ ; hence, $c_2 = 0$ , since the number $v_2^{\mathrm{T}} v_2 \neq 0$ . Multiply (*) by $v_3^{\mathrm{T}}$ to conclude that $c_3 = 0$ . We have shown that each $c_i$ MUST be zero. We conclude that $v_1$ , $v_2$ , and $v_3$ are linearly independent.