Math 544, Final Exam, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 14 problems. Problem 1 is worth 22 points. Each of the rest of the problems is worth 6 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website shortly after the exam is finished.

1. Let
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$
   (a) Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.
   (b) Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.
   (c) Find a basis for the null space of A.
   (d) Find a basis for the column space of A.
   (e) Find a basis for the row space of A.
   (f) Express each column of A in terms of your answer to (d).
   (g) Express each row of A in terms of your answer to (e).

2. Find an orthogonal basis for the null space of the matrix A from problem 1.

3. Find a matrix B with $B^2 = A$, where $A = \begin{bmatrix} -5 & -9 \\ 6 & 10 \end{bmatrix}$.

4. Define "linearly dependent". Use complete sentences.

5. Define "linear transformation". Use complete sentences.

6. Define "null space". Use complete sentences.

7. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which fixes the origin and rotates the xy-plane counter-clockwise by 45 degrees. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

8. Let A be a matrix, v_1 and v_2 be non-zero vectors, and λ_1 and λ_2 be real numbers. Suppose that $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, and $\lambda_1 \neq \lambda_2$. PROVE that v_1 and v_2 are linearly independent.

9. Let $T\colon V \to W$ be a linear transformation, and let v_1 , v_2 , and v_3 be vectors in V , with $T(v_1)$, $T(v_2)$, and $T(v_3)$ linearly independent. Do the vectors v_1 , v_2 , and v_3 have to be linearly independent? If yes, PROVE the result. If no, give a counter EXAMPLE.

10. Let V be the vector space of all polynomials $p(x)$ of degree three or less which have the property that $p(2) = 0$ and $p'(2) = 0$. Find a basis for V . Explain.

11. Let V be the vector space of all differentiable real-valued functions which are defined on all of $\mathbb{R}$. Let W be the vector space of all real-valued functions which are defined on all of $\mathbb{R}$. Let T from V to W be the function which is given by $T(f(x)) = f'(x)$. Is T a linear transformation? Explain very thoroughly.

12. Let A and B be $n \times n$ matrices. Is the null space of B contained in the null space of AB ? If yes, PROVE the result. If no, give a counter EXAMPLE.

13. Let A and B be $n \times n$ matrices. Is the column space of B contained in the column space of AB ? If yes, PROVE the result. If no, give a counter EXAMPLE.

14. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \notin V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V ? Explain very thoroughly.