**Math 544, Exam 4 Solutions, Summer 2004**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 6 problems. Problems 1 and 2 are worth 9 points each. Each of the other problems is worth 8 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Express** $v = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$ **as a linear combination of** $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ **,** $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ **,** **and** $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ **. (It might be helpful to notice that** $v_1$ **,** $v_2$ **and** $v_3$ **are an orthogonal set.)**

Multiply each side of $v = c_1v_1 + c_2v_2 + c_3v_3$ by $v_1^{\mathrm{T}}$ to see that $27v = 3c_1$ , so $c_1 = 9$ . Similar calculations give $-1 = 2c_2$ ; so, $c_2 = -\frac{1}{2}$ ; and $-3 = 6c_3$ ; so, $c_3 = -\frac{1}{2}$ . We check that

$$\boxed{9v_1 - \tfrac{1}{2}v_2 - \tfrac{1}{2}v_3 = 9\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix} = v.}$$

2. **Express** $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **as a linear combination of** $M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **,** $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ **,** $M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **, and** $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **.**

Observe that

$$\boxed{2M_1 - M_2 + 2M_3 + M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = M.}$$

3. **List a basis for the vector space of $3 \times 3$ skew symmetric matrices. (No proof is needed.) Recall that the matrix $M$ is skew-symmetric if $M^{\mathrm{T}} = -M$.**

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}$$

4. **Let $W$ be the set of all continuous functions $f(x)$ with the property that $\int_0^1 f(x)dx = 0$. Is $W$ a vector space? Explain.**

$\boxed{\text{YES}}$

$W$ **is closed under addition.** If $f$ and $g$ are in $W$, then $f+g$ is in $W$ because

$$\int_0^1 [f(x)+g(x)]dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0.$$

$W$ **is closed under scalar multiplication.** If $f$ is in $W$ and $r$ is a number, then $rf$ is in $W$ because

$$\int_0^1 rf(x)dx = r\int_0^1 f(x)dx = r \cdot 0 = 0.$$

**The zero function is in** $W$ because $\int_0^1 0dx = 0$.

5. **Let $W$ be the set of all twice differentiable functions $f(x)$ with the property that $f''(x) + f(x) = e^x$. Is $W$ a vector space? Explain.**

$\boxed{\text{NO!}}$ The set $W$ is not closed under scalar multiplication. Indeed, $f(x) = \frac{1}{2}e^x$ is in $W$ (because $\frac{1}{2}e^x + \frac{1}{2}e^x = e^x$). But $6f(x) = 3e^x$ is not in $W$ because $3e^x + 3e^x = 6e^x \neq e^x$.

6. **Find an orthogonal basis for the null space of $A = [1 \quad 3 \quad 4 \quad 5]$.**

One basis for the null space of $A$ is

$$v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let

$$u_1 = v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{12}{10} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \left( \begin{bmatrix} -20 \\ 0 \\ 5 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 5u_2' = \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

We check that $u_1^{\mathrm{T}} u_2 = 0$ and $Au_2 = 0$. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{15}{10} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{10}{65} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{13} \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{26} \left( \begin{bmatrix} -130 \\ 0 \\ 0 \\ 26 \end{bmatrix} - 39 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{26} \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.$$

Let

$$u_3 = 26u_3' = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.$$

Check that $Au_3 = 0$, $u_1^{\mathrm{T}} u_3 = 0$, and $u_2^{\mathrm{T}} u_3 = 0$. Our answer is

$$\boxed{u_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.}$$