Math 544, Exam 3, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 9 problems. Problem 1 is worth 10 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Let**

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 12 \\ 1 & 2 & 2 & 1 & 16 \\ 1 & 2 & 2 & 2 & 21 \\ 3 & 6 & 5 & 4 & 49 \end{bmatrix}.$$

   (a) **Find a basis for the null space of A.**
   (b) **Find a basis for the column space of A.**
   (c) **Express each column of A as a linear combination of the vectors in your answer to (b).**
   (d) **What is the dimension of the null space of A?**
   (e) **What is the dimension of the column space of A?**

Apply $R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - R_1$ and $R_4 \mapsto R_4 - 3R_1$ to get

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 12 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 1 & 13 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 - R_2$, $R_3 \mapsto R_3 - R_2$ and $R_4 \mapsto R_4 - 2R_2$ to get

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 8 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 - R_3$ and $R_4 \mapsto R_4 - R_3$ to get

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) A basis for the null space of A is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{bmatrix}.$$

(b) A basis for the column space of A is

$$A_{*,1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad A_{*,3} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

(c) $A_{*,2} = 2A_{*,1}$, and $A_{*,5} = 3A_{*,1} + 4A_{*,3} + 5A_{*,4}$. (I use $A_{*,j}$ to mean column j of A.)
(d) The null space of A has dimension 2.
(e) The column space of A has dimension 3.

2. **Define "span". Use complete sentences.**

The vectors $v_1, v_2, \ldots, v_n$ in the vector space V *span* V if every vector in V is equal to a linear combination of $v_1, v_2, \ldots, v_n$.

3. **Define "basis". Use complete sentences.**

A *basis* for the vector space V is a set of vectors in V which span V and are linearly independent.

4. **Let U and V be subspaces of $\mathbb{R}^n$ and let**

$$W = \{w \in \mathbb{R}^n \mid w = u + v \textbf{ for some } u \in U \textbf{ and } v \in V\}.$$

Prove that W is a subspace of $\mathbb{R}^n$.

The set W is closed under addition. Take w_1 and w_2 from W. Well, $w_1 = u_1 + v_1$ and $w_2 = u_2 + v_2$ for some $u_i \in U$ and $v_i \in V$.We see that

$$w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2);$$

furthermore, $u_1 + u_2 \in U$ because U is a vector space and $v_1 + v_2$ is in V because V is a vector space. We conclude that $w_1 + w_2$ is equal to an element of U plus an element of V; and therefore, $w_1 + w_2$ is in W.

The set W is closed under scalar multiplication. Take $w_1 = u_1 + v_1 \in W$, as above, and $r \in \mathbb{R}$. We see that $rw_1 = ru_1 + rv_1$. The vector space U is closed under scalar multiplication; so, ru_1 is in U. Also, rv_1 is in V again because V is a vector space. Once again rw_1 has the correct form; that is rw_1 is equal to an element of U plus an element of V; therefore, rw_1 is in W.

The zero vector in $\mathbb{R}^m$ is equal to the zero vector of U plus the zero vector of V; and therefore, the zero vector is in W.

5. **Let V be the set of all polynomials $f(x)$, such that $f(x)$ has real number coefficients, $f(x)$ has degree at most 4, and $f(1)=0$. Is V a vector space? Explain fairly thoroughly.**

YES.
The set V is closed under addition. Take $f(x)$ and $g(x)$ in V. It is clear that $f(x)+g(x)$ is a polynomial of degree at most 4. It is also clear that if we plug 1 in for x, then the answer is $f(1)+g(1)=0+0=0$; thus, $f(x)+g(x)$ is in V.
The set V is closed under scalar multiplication. Take $f(x)\in V$ and $r\in\mathbb{R}$. It is clear that $rf(x)$ is a polynomial of degree at most 4. It is also clear that $rf(1)=r(0)=0$. Thus, $rf(x)\in V$.
The zero polynomial sends 1 to 0, so the zero polynomial is in V.

6. **Let V be the set of all 2×2 non-singular matrices. Is V a vector space? Explain fairly thoroughly.**

NO. The zero matrix is not in V.

7. **Let A and B be $n\times n$ matrices. Is the column space of AB always contained in the column space of B? If yes, give a proof. If no, give an example.**

NO. Let $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&0\end{bmatrix}$. We see that $AB=\begin{bmatrix}0&0\\1&0\end{bmatrix}$. The vector $v=\begin{bmatrix}0\\1\end{bmatrix}$ is in the column space of AB, but v is not a multiple of $\begin{bmatrix}1\\0\end{bmatrix}$; so, b is not in the colmun space of B.

8. **Let A and B be $n\times n$ matrices. Is the null space of AB always contained in the null space of B? If yes, give a proof. If no, give an example.**

NO. Let A be the zero matrix and B be the identity matrix. The product AB is the zero matrix; hence, the null space of AB is all of $\mathbb{R}^n$. The null space of B is $\{0\}$; and $\mathbb{R}^n$ is not contained in $\{0\}$.

9. **Let**

$$v_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\quad v_2=\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\quad v_3=\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\quad v_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1\in V$, $v_2\in V$, $v_3\in V$, and $v_4\notin V$. Do you have enough information to determine the dimension of V? Explain thoroughly.

The vector space V has dimension 3. We have exhibited 3 linearly independent vectors v_1, v_2 and v_3 in V. So $\dim V\geq 3$. On the other hand, V is a subspace of the 4 dimensional vector space $\mathbb{R}^4$; so $\dim V\leq 4$. Finally, if $\dim V$ were equal to 4; then V would have to equal $\mathbb{R}^4$. However, V does not equal $\mathbb{R}^4$ because v_4 is not in V.