Exam 3, Summer 2003, Math 544, Solutions

PRINT Your Name:______________________________

Please also write your name on the back of the exam.

There are 9 problems on 5 pages. Problem 4 is worth 10 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door later today (surely by 5:00 PM), you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define "linearly independent". Use complete sentences.**
The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = 0$, $c_2 = 0$, $\dots$, $c_p = 0$.

2. **Define "null space". Use complete sentences.**
The *null space* of the matrix A is the set of all column vectors x with $Ax = 0$.

3. **Define "span". Use complete sentences.**
The vectors $v_1, \dots, v_p$ *span* the vector space V is every vector in V is equal to a linear combination of the vectors $v_1, \dots, v_p$.

4. **Let** $A = \begin{bmatrix} 1 & 2 & 3 & 3 & 6 & 7 \\ 1 & 2 & 3 & 3 & 6 & 8 \\ 2 & 4 & 6 & 6 & 12 & 15 \\ 1 & 2 & 3 & 4 & 11 & 1 \end{bmatrix}$. **Find a basis for the null space of** A**. Find a basis for the column space of**

A. Find a basis for the row space of A. Express each column of A as a linear combination of the basis you have chosen for the column space of A. Express each row of A as a linear combination of the basis you have chosen for the row space of A.

Apply $R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - 2R_1$, and $R_4 \mapsto R_4 - R_1$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 & -6 \end{bmatrix}.$$

Exchange rows 2 and 4 to get:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 6 & 7 \\ 0 & 0 & 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 - 7R_3$, $R_2 \mapsto R_2 + 6R_3$, and $R_4 \mapsto R_4 - R_3$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 - 3R_2$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & -9 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The null space of A is the set of all vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 9 \\ 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix},$$

where the "free variables" x_2, x_3, and x_5 are free to taken on any values. It is now obvious that

$$\boxed{\begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-3\\0\\1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}9\\0\\0\\-5\\1\\0\end{bmatrix}}$$

is a basis for the null space of A. (These vectors are in the null space of A. Be sure to check this. Every vector in the null space of A can be written in terms of these vectors. A quick glance shows that these vectors are linearly independent.)

Columns 1, 4, and 6 of the original matrix A are a basis for the column space of A. That is

$$\boxed{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \quad \begin{bmatrix}3\\3\\6\\4\end{bmatrix}, \quad \begin{bmatrix}7\\8\\15\\1\end{bmatrix}}$$

are a basis for the column space of A. Look at the basis of the null space to see that

$$A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,5} = -9A_{*,1} + 5A_{*,4}.$$

Be sure to check that these equations hold.

The vectors

$$\boxed{\begin{bmatrix}1 & 2 & 3 & 0 & -9 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 1 & 5 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}}$$

are a basis for the row space of A. Notice that

$$A_{1,*} = 1\begin{bmatrix}1 & 2 & 3 & 0 & -9 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 0 & 0 & 1 & 5 & 0\end{bmatrix} + 7\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$A_{2,*} = 1\begin{bmatrix}1 & 2 & 3 & 0 & -9 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 0 & 0 & 1 & 5 & 0\end{bmatrix} + 8\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$A_{3,*} = 2\begin{bmatrix}1 & 2 & 3 & 0 & -9 & 0\end{bmatrix} + 6\begin{bmatrix}0 & 0 & 0 & 1 & 5 & 0\end{bmatrix} + 15\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

and

$$A_{4,*} = 1\begin{bmatrix}1 & 2 & 3 & 0 & -9 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 0 & 0 & 1 & 5 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

Do be sure to check this arithmetic. I use $A_{i,*}$ to mean row i of A and $A_{*,j}$ to mean column j of A.

5. **Let A and B be 2×2 matrices. Does**

$$\textbf{the column space of } AB \subseteq \textbf{the column space of } A$$

always happen? If yes, prove it. If no, give an example. $\boxed{\text{yes}}$. Take x in the column space of AB. So $x = ABy$ for some vector $y \in \mathbb{R}^2$. It follows that $x = A(By)$ and By is a vector in $\mathbb{R}^2$ so x is also in the column space of A.

6. **Let A and B be 2×2 matrices. Does**

$$\textbf{the null space of } AB \subseteq \textbf{the null space of } A$$

always happen? If yes, prove it. If no, give an example. $\boxed{\text{no}}$. Let A be the identity matrix and B be the zero matrix. In this case AB is the zero matrix. We see that $\begin{bmatrix}1\\0\end{bmatrix}$ is in the null space of $AB = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, but $\begin{bmatrix}1\\0\end{bmatrix}$ is not in the null space of $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

7. **True or False. Let** $W = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| x_2x_3 = 0 \right\}$**. Is W a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.**

$\boxed{\text{no}}$. We see that $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are both in W, but $v + v' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in W.

8. **True or False. Let** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + x_2 = x_3 \right\}$**. Is** W **a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.**

$\boxed{\text{yes}}$. We see that W is the null space of the matrix $[1 \quad 1 \quad -1]$ and we know that the null space of any matrix is a vector space.

9. **Let** A **be a** 2×3 **matrix. Suppose that the column space of** A **has dimension** 2**. Is the system of equations** $Ax = b$ **consistent for every choice of the vector** b **in** $\mathbb{R}^2$ **? Explain.**

$\boxed{\text{yes}}$. The column space of A is a two dimensional subspace of $\mathbb{R}^2$, which has dimension 2. It follows that the column space of A is equal to $\mathbb{R}^2$. In other words, if b is a vector in $\mathbb{R}^2$, then there exists a vector x in $\mathbb{R}^3$, with $Ax = b$. That, is $Ax = b$ has a solution for every choice of b.