Math 544, Summer 2003, Exam 2

PRINT Your Name:________________________________

Please also write your name on the back of the exam.

There are 10 problems on 5 pages. Each problem is worth 5 points. The exam is worth a total of 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door later today, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define "linearly independent". Use complete sentences.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

2. **Define "non-singular". Use complete sentences.**

The square matrix A is *non-singular* if the only column vector x with $Ax = 0$ is $x = 0$.

3. **Let A be an $n \times n$ matrix. List three conditions which are equivalent to the statement "A is non-singular". (I expect three new conditions in addition to "A is non-singular". Also, I do not expect you to repeat your answer to problem 2.)**

The following conditions are equivalent.

(0) The matrix A is non-singular.

(1) The columns of A are linearly independent.

(2) The system of equations $Ax = b$ has a unique solution for each vector $b \in \mathbb{R}^n$.

(3) The matrix A is invertible.

4. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{rrrrrcl} x_1 & +3x_2 & +4x_3 & +2x_4 & +4x_5 & = & 16 \\ x_1 & +3x_2 & +4x_3 & +3x_4 & +6x_5 & = & 21 \\ 2x_1 & +6x_2 & +8x_3 & +5x_4 & +10x_5 & = & 37 \end{array}$$

Consider

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 2 & 4 & 16 \\ 1 & 3 & 4 & 3 & 6 & 21 \\ 2 & 6 & 8 & 5 & 10 & 37 \end{array}\right]$$

Apply $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 2 & 4 & 16 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Apply $R_3 \mapsto R_3 - R_2$ and $R_1 \mapsto R_1 - 2R_2$ to get:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution of this system of equations is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.}$$

Some specific solutions are:

$$\begin{bmatrix} 6 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

(We took $x_2 = x_3 = x_5 = 0$, $x_2 = 1$ with $x_3 = x_5 = 0$, $x_3 = 1$ with $x_2 = x_5 = 0$, $x_5 = 1$ with $x_2 = x_3 = 0$.) We check that each specific solution does indeed satisfy the equations:

$$\begin{array}{ccc} 6+10=16 & 3+3+10=16 & 2+4+10=16 \\ 6+15=21 & 3+3+15=21 & 2+4+15=21 \\ 12+25=37 & 6+6+25=37 & 4+8+25=37 \end{array}$$

$$\begin{array}{c} 6+6+4=16 \\ 6+9+6=21 \\ 12+15+10=37.\checkmark \end{array}$$

5. **Are the vectors** $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$**,** $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$**,** $v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ **linearly independent? Explain.**

We solve

$$(*) \qquad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

We look at

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

Apply $R_2 \mapsto R_2 - 2R_1$ and $R_3 \mapsto R_3 - 3R_1$ to get

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Apply $R_3 \mapsto R_3 - 2R_2$ to get

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Replace $R_2 \mapsto -(1/3)R_2$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \mapsto R_1 - 4R_2$ to get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of (*) is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Take $c_3 = 1$ to get

$$1v_1 - 2v_2 + 1v_3 = 0.$$

Do check that this claim is true:

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a non-trivial linear combination of v_1, v_2, v_3 which equals zero. We conclude that v_1, v_2, v_3 are $\boxed{\text{linearly dependent.}}$

6. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B are 2×2 invertible matrices, then AB is an invertible matrix.**

$\boxed{\text{TRUE.}}$ If the inverse of A is called A^{-1} and the inverse of B is called B^{-1}, then $B^{-1}A^{-1}$ is the inverse of AB because $B^{-1}A^{-1}AB = I$ and $ABB^{-1}A^{-1} = I$.

7. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If the vectors v_1 , v_2 , and v_3 are linearly independent, then the vectors v_1+v_3 , v_2+v_3 , and v_1+v_2 are also linearly independent.**

TRUE. Suppose

$$(**) \qquad c_1(v_1+v_3)+c_2(v_2+v_3)+c_3(v_1+v_2)=0.$$

Then,

$$(c_1+c_3)v_1+(c_2+c_3)v_2+(c_1+c_2)v_3=0.$$

The vectors v_1, v_2, v_3 are linearly independent; consequently, the coefficients c_1+c_3 , c_2+c_3 , c_1+c_2 all are zero. It will take a few minutes to find the solution set of the equations

$$\begin{aligned} c_1+c_3&=0\\ c_2+c_3&=0\\ c_1+c_2&=0. \end{aligned}$$

Fortunately, we are well prepared for that project. Look at

$$\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0 \end{bmatrix}.$$

Apply $R_3 \mapsto R_3 - R_1$ to get

$$\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 1 & -1 \end{bmatrix}.$$

Apply $R_3 \mapsto R_3 - R_2$ to get

$$\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & -2 \end{bmatrix}.$$

The bottom equation tells me that c_3 must be zero. The second equation now tells me that c_2 must be zero. The top equation now tells me that $c_1 = 0$. The only solution of (**) is $c_1 = c_2 = c_3 = 0$. We conclude that $v_1 + v_3$, $v_2 + v_3$, and $v_1 + v_2$ are linearly independent.

8. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If the vectors v_1, v_2, and v_3 are linearly independent, then the vectors $v_1 - v_3$, $v_3 - v_2$, and $v_2 - v_1$ are also linearly independent.**

FALSE. It does not matter what v_1, v_2, and v_3 are, we have

$$1(v_1 - v_3) + 1(v_3 - v_2) + 1(v_2 - v_1) = 0.$$

Thus, we have a non-trivial linear combination of $v_1 - v_3$, $v_3 - v_2$, and $v_2 - v_1$ which equals zero. We conclude that $v_1 - v_3$, $v_3 - v_2$, and $v_2 - v_1$ are linearly dependent. If you want to make a concrete counterexample, take

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We see that v_1, v_2, and v_3 are linearly independent, but

$$v_1 - v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_3 - v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 - v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly dependent because

$$1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

9. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B, and C are 2×2 matrices, with $A \neq 0$ and $BA = CA$, then $B = C$.**

FALSE. Take $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 45 \\ 0 & 86 \end{bmatrix}$. We see that $A \neq 0$, $B \neq C$, but BA and CA are both equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

10. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B are 2×2 matrices, with AB equal to the identity matrix, then BA is also equal to the identity matrix.**

TRUE. The hypothesis $AB = I$ ensures that B is non-singular, because if $Bx = 0$, then

$$x = Ix = ABx = A0 = 0.$$

The non-singular matrix theorem tells us that B is invertible. (See problem 3.) Let B^{-1} be the inverse of B. So, $BB^{-1} = I$ and $B^{-1}B = I$. Now multiply $AB = I$ by B^{-1} (on the right) to get

$$A = ABB^{-1} = IB^{-1} = B^{-1}.$$

So, $A = B^{-1}$. We see that $BA = BB^{-1} = I$.