SOLUTIONS to Exam 4, Math 544, Spring, 2003

PRINT Your Name:__________________________________

Please also write your name on the back of the exam.

There are 8 problems on 4 pages. Problem 1 is worth 15 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail.

I will leave your exam outside my office door about 6 PM SATURDAY, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the exam is finished.

1. **Find a matrix B with $B^2 = A$ for $A = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}$. I expect you to write down the four entries of B. Check your answer.**

The eigenvalues of A are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 13-\lambda & 18 \\ -6 & -8-\lambda \end{bmatrix}$$

$$= (13-\lambda)(-8-\lambda) - (18)(-6) = \lambda^2 - 5\lambda - 104 + 108 = \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 4)(\lambda - 1).$$

The eigenvalues of A are $\lambda = 4$ and $\lambda = 1$. The eigenspace which belongs to $\lambda = 1$ is the null space of

$$A - I = \begin{bmatrix} 12 & 18 \\ -6 & -9 \end{bmatrix}.$$

Divide row 1 by 12 to get $\begin{bmatrix} 1 & 3/2 \\ -6 & -9 \end{bmatrix}$. Add 6 copies of row 1 to row 2 to get $\begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$. The eigenspace of A which belongs to 1 is the set of all vectors x with $x_1 = -\frac{3}{2}x_2$, and x_2 is arbitrary. The vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ belongs to $\lambda = 1$. We check this statement:

$$A\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -39+36 \\ +18-16 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \checkmark$$

The eigenspace which belongs to $\lambda = 4$ is the null space of

$$A - 4I = \begin{bmatrix} 9 & 18 \\ -6 & -12 \end{bmatrix}.$$

Divide row 1 by 9 to get $\begin{bmatrix} 1 & 2 \\ -6 & -12 \end{bmatrix}$. Add six copies of row 1 to row two to get $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ belongs to $\lambda = 4$. We check this statement:

$$A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -26+18 \\ 12-8 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$= 4\begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark.$$

We now see that

$$A\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$. We see that $S^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, and that $A = SDS^{-1}$. Our answer is

$$B = S\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}S^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}.}$$

Check:

$$B^2 = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix} = A. \checkmark$$

2. **Define "linear transformation". Use complete sentences.**

Let V and W be vector spaces. A function T from V to W is called a *linear transformation* if
$T(v_1 + v_2) = T(v_1) + T(v_2)$ and
$T(rv_1) = rT(v_1)$
for all $r \in \mathbb{R}$ and $v_1, v_2 \in V$.

3. **Define "eigenvalue". Use complete sentences.**

Let A be a square matrix. The number λ is an *eigenvalue* of A if there exists a non-zero vector v with $Av = \lambda v$.

4. **Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = -x$. Find a matrix A with $T(v) = Av$ for all $v \in \mathbb{R}^2$. Check your answer.**

We saw in class that T is a linear transformation. It follows that the first column of A is $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, which is equal to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. (Draw a picture.) The second column of A is $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, which is equal to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. (Draw a picture.) So,

$$\boxed{A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.}$$

Check. The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is on the line $y = -x$ and

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

just like it should. Also, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is on the line $y = x$. Reflection acts like multiplication by minus one on this line, and

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

just like we expected.

5. **Prove that every 3×3 skew-symmetric matrix is singular. (Recall that the matrix A is *skew-symmetric* if $A^{\mathrm{T}} = -A$.)**

We know that

$$\det A = \det(A^{\mathrm{T}}) = \det(-A) = (-1)^3 \det A = -\det A.$$

The first equality holds for all matrices. The second equality holds because A is skew-symmetric. The third equality holds because we had to pull minus one out from each of three rows.

Add $\det A$ to both sides to get $2 \det A = 0$. Divide by two to see $\det A = 0$. Conclude that A is singular.

6. **Prove that every 2×2 symmetric matrix has at least one real eigenvalue. (Recall that the matrix A is *symmetric* if $A^{\mathrm{T}} = A$.)**

Write $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. The eigenvalues of A are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ b & d - \lambda \end{bmatrix} = (a - \lambda)(d - \lambda) - b^2$$

$$= \lambda^2 + (-a-d)\lambda + ad - b^2.$$

The quadratic formula tells us that the solutions of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0$$

are

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

In our problem, $A = 1$, $B = -a - d$ and $C = ad - b^2$. The eigenvalues of A are

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2}$$

$$= \frac{a + d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2}$$

$$= \frac{a + d \pm \sqrt{a^2 - 2ad + d^2 + 4b^2}}{2}$$

$$= \frac{a + d \pm \sqrt{(a-d)^2 + 4b^2}}{2}.$$

Notice that the number under the radical is the SUM of two perfect squares. This number is ZERO or HIGHER. So, when we take the square root, we will get real number answers (rather than complex numbers which are not real).

7. **Let v_1, v_2, and v_3 be an orthogonal set of non-zero vectors. Prove that v_1, v_2, and v_3 are linearly independent.**

Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Multiply by v_1^{T}, to see that $c_1v_1^{\mathrm{T}}v_1 = 0$. I know that the number $v_1^{\mathrm{T}}v_1$ is not zero because the vector v_1 is not zero. Conclude that the number c_1 must be zero. Multiply the equation by v_2^{T} to learn that c_2 must be zero, and v_3^{T} to learn that c_3 must be zero.

8. **True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If A and B are square matrices, then the null space of $A + B$ is contained in the intersection of the null space of A and the null space of B.**

This statement is $\boxed{\text{FALSE}}$. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Notice that $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The null space of $A + B$ is $\mathbb{R}^2$. The null space of A is $\{0\}$. The null space of B is $\{0\}$. It is certainly not true that $\mathbb{R}^2 \subseteq \{0\} \cap \{0\}$.