Math 544, Exam 1, Spring 2003, SOLUTIONS

PRINT Your Name:________________________________

Please also write your name on the back of the exam.

There are 10 problems on 6 pages. Each problem is worth 5 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you.

I will leave your exam outside my office door about 6PM today, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the exam is finished.

1. **Find the general solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{lll} 2x_1 & +3x_2 = & 4 \\ 4x_1 & +6x_2 = & 7 \end{array}$$

Start with $\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 6 & 7 \end{array}\right]$. Replace $R_2 \mapsto R_2 - 2R_1$ to get $\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & -1 \end{array}\right]$. The bottom row tells us that our solution must satisfy $0 = -1$. This does not happen.

$$\boxed{\text{The system of equations has NO solution.}}$$

2. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{llllll} x_1 & +3x_2 & & +3x_4 & = & 1 \\ x_1 & +3x_2 & +x_3 & +5x_4 & = & 3 \\ 2x_1 & +6x_2 & +x_3 & +8x_4 & = & 4. \end{array}$$

Start with $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 1 & 5 & 3 \\ 2 & 6 & 1 & 8 & 4 \end{array}\right]$. Replace $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to get $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$. Replace $R_3 \mapsto R_3 - R_2$ to get $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The GENERAL solution is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.}$$

We record three specific examples. In the first, we take $x_2 = x_4 = 0$. In the second, we take $x_2 = 1$ and $x_4 = 0$. In the third, we take $x_2 = 0$ and $x_4 = 1$. Our SPRCIFIC solutions are:

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.}$$

These solutions work because

$$\begin{array}{ccc} 1 = 1 & -2+3 = 1 & -2+3 = 1 \\ 1+2 = 3 & -2+3+2 = 3 & -2+5 = 3 \\ 2+2 = 4 & -4+6+2 = 4 & -4+8 = 4. \end{array}$$

3. **Express** $b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ **as a linear combination of** $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **and** $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$**.** We must find c_1 and c_2 with $c_1v_1 + c_2v_2 = b$.

We apply Guassian Elimination to $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 4 & 8 \end{array}\right]$. Replace $R_2 - 2R_1$ to get $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2 & -2 \end{array}\right]$. Replace $R_2 \mapsto (-1/2)R_2$ to get $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$. Replace $R_1 \mapsto R_1 - 3R_2$ to get $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$. We see that $c_1 = 2$ and $c_2 = 1$. We conclude that

$$\boxed{b = 2v_1 + v_2},$$

and of course, this is correct because

$$2v_1 + v_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = b.\checkmark$$

4. **Define "linear combination". Use complete sentences.** The vector v is a <u>linear combination</u> of the vectors $v_1, \dots, v_p$, if there are numbers $c_1, \dots, c_p$, with $v = c_1 v_1 + \dots + c_p v_p$.

5. **Define "linearly independent". Use complete sentences.** The vectors $v_1, \dots, v_p$ are <u>linearly independent</u> if the only numbers $c_1, \dots, c_p$, with $c_1 v_1 + \dots + c_p v_p = 0$, are $c_1 = \dots = c_p = 0$.

6. **STATE the theorem about linear dependence or independence of p vectors in $\mathbb{R}^m$. (I call this theorem the Short Fat Theorem.)** If $m < p$, then every set of p vectors in $\mathbb{R}^m$ is linearly dependent.

7. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B, and C are 2×2 matrices with $A \neq 0$ and $BA = CA$, then $B = C$.** FALSE. Here is a counterexample. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$. Notice that $A \neq 0$, $B \neq C$ and $BA = CA$ because

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$CA = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

8. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B are 2×2 symmetric matrices, then AB is a symmetric matrix.** FALSE. Here is a counterexample. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A and B are each symmetric but

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

is not symmetric.

9. **Which numbers a make $A = \begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}$ non-singular? Explain.** The matrix A is non-singular if the only column vector x with $Ax = 0$ is the zero column vector. We solve $Ax = 0$ and interpret our answer. We apply Gaussian Elimination to $\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}$. (In our heads we store the augmented column which consists entirerly of zeros throughout the entire calculation!) Replace $R_2 \mapsto R_2 - 2R_1$ to get $\begin{bmatrix} 1 & 2 \\ 0 & a-4 \end{bmatrix}$. This is far enough. If $a - 4$ is equal to zero, then $Ax = 0$ has an infinite number of solutions. On the other hand, if $a - 4$ is

not equal to zero, then the present matrix showes us that x_2 must be zero and then x_1 must be zero.

$$\boxed{\text{The matrix } A \text{ is non-singular for every choice of } a, \text{ except } a = 4.}$$

10. **Let** v_1 **,** v_2 **,** v_3 **be linearly independent vectors. PROVE that the vectors** $v_1 + v_2$ **,** $v_1 - v_2$ **,** v_3 **are also linearly independent.** Suppose that c_1, c_2, c_3 are numbers with

$$c_1(v_1 + v_2) + c_2(v_1 - v_2) + c_3 v_3 = 0.$$

We must show that c_1 , c_2 , and c_3 all must be zero. Re-write the most recent equation as

$$(c_1 + c_2)v_1 + (c_1 - c_2)v_2 + c_3 v_3 = 0.$$

We are told that the vectors v_1 , v_2 , and v_3 are linearly independent. It follows that

$$c_1 + c_2 = 0, \quad c_1 - c_2 = 0, \quad \text{and} \quad c_3 = 0.$$

Add the first two equations to see that $2c_1 = 0$. Divide by 2 to see that $c_1 = 0$. The first equation now tells us that $c_2 = 0$. The third equation tells us that $c_3 = 0$. We have shown that the only numbers c_1, c_2, c_3 with

$$c_1(v_1 + v_2) + c_2(v_1 - v_2) + c_3 v_3 = 0$$

are $c_1 = c_2 = c_3 = 0$. We conclude that the vectors $v_1 + v_2$, $v_1 - v_2$, v_3 are linearly independent.