Quiz 8 Math 544, Monday, November 9, 2020

Let U and V be finite dimensional subspaces of the vector space W. Recall that $U\cap V$ and $U+V$ are the vector spaces

$$U\cap V=\{w\in W \mid w\in U \text{ and } w\in V\} \quad \text{and}$$

$$U+V=\{w\in W \mid \text{ there exists } u\in U \text{ and } v\in V \text{ with } w=u+v\}.$$

Give a formula which relates the following vector space dimenions $\dim U$, $\dim V$, $\dim(U\cap V)$ and $\dim(U+V)$. Give a **complete and correct** proof of your formula.

The formula is

$$\dim(U+V)=\dim U+\dim V-\dim(U\cap V).$$

Proof. Let $w_1,\dots,w_r$ be a basis for $U\cap V$. The vectors $w_1,\dots,w_r$ are linearly independent in the vector space U; so there are elements $u_1,\dots,u_s$ in U so that $w_1,\dots,w_r,u_1,\dots,u_s$ is a basis for U. In a similar manner, the vectors $w_1,\dots,w_r$ are linearly independent in the vector space V; so there are elements $v_1,\dots,v_t$ in V so that $w_1,\dots,w_r,v_1,\dots,v_t$ is a basis for V. We will prove that the vectors

$$w_1,\dots,w_r,u_1,\dots,u_s,v_1,\dots,v_t \quad \text{form a basis for } U+V. \tag{1}$$

Once we do this then

$$\dim(U+V)=r+s+t,\quad \dim U=r+s,\quad \dim V=r+t,\quad \dim(U\cap V)=r,$$

and

$$\dim U+\dim V-\dim(U\cap V)=(r+s)+(r+t)-r=r+s+t=\dim(U+V).$$

We prove (1).

It is clear that $w_1,\dots,w_r,u_1,\dots,u_s,v_1,\dots,v_t$ span $U+V$. Indeed, every element of $U+V$ has the form $u+v$ for some $u\in U$ and some $v\in V$. On the other hand, u can be written in terms of $w_1,\dots,w_r,u_1,\dots,u_s$ and v can be written in terms of $w_1,\dots,w_r,v_1,\dots,v_t$. It follows that $u+v$ can be written in terms of $w_1,\dots,w_r,u_1,\dots,u_s,v_1,\dots,v_t$.

Now we prove that $w_1,\dots,w_r,u_1,\dots,u_s,v_1,\dots,v_t$ are linearly independent. Suppose there are numbers a_i, b_j, and c_k such that

$$\sum_{i=1}^{r}a_iw_i+\sum_{j=1}^{s}b_ju_j+\sum_{k=1}^{t}c_kv_k=0. \tag{2}$$

The sum

$$\sum_{i=1}^{r}a_iw_i+\sum_{j=1}^{s}b_ju_j=-\sum_{k=1}^{t}c_kv_k$$

is in $U \cap V$. The vectors $w_1, \ldots, w_r$ are a basis for $U \cap V$; hence, there are numbers $d_1, \ldots, d_r$ so that

$$-\sum_{k=1}^{t} c_k v_k = \sum_{i=1}^{r} d_i w_i.$$

However, the vectors $w_1, \ldots, w_r, v_1, \ldots, v_t$ are linearly independent; hence, $c_1, \ldots, c_t$ are all zero!

Now equation (2) says that

$$\sum_{i=1}^{r} a_i w_i + \sum_{j=1}^{s} b_j u_j = 0.$$

The vectors $w_1, \ldots, w_r, u_1, \ldots, u_s$ are linearly independent; thus each a_i and each b_j is zero. We conclude that $w_1, \ldots, w_r, u_1, \ldots, u_s, v_1, \ldots, v_t$ are linearly independent. The proof is complete. $\square$