Math 544, Final Exam, Fall 2009

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 200 points. There are **15** problems. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (13 points) Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.

2. (13 points) Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.

3. (13 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.

4. (13 points) Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.

5. (13 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation of vector spaces with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}.$$

   Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$. **Check your answer.**

6. (13 points) Let U_1 and U_2 be subspaces of the vector space V. Does the union $U_1 \cup U_2$ have to be a vector space? If yes, prove it. If no, give an example. (Recall that the vector u is in $U_1 \cup U_2$ if u is in U_1 OR u is in U_2.)

7. (13 points) Let U_1 and U_2 be subspaces of the vector space V. Does the intersection $U_1 \cap U_2$ have to be a vector space? If yes, prove it. If no, give an example. (Recall that the vector u is in $U_1 \cap U_2$ if u is in U_1 AND u is in U_2.)

8. (13 points) Give an example of a matrix M for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector belonging to the eigenvalue 1 and $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is an eigenvector belonging to the eigenvalue 2. **Check your answer.**

9. (13 points) Suppose that A is a matrix with distinct eigenvalues λ_1, λ_2, and λ_3. Suppose further that v_1, v_2, and v_3 are nonzero eigenvectors of A with v_i belonging to λ_i. Prove that v_1, v_2, and v_3 are linearly independent.

10. (13 points) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3 is a basis for W. Suppose further that w_1 and w_2 are in V, but w_3 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

11. (13 points) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3, w_4 is a basis for W. Suppose further that w_1 and w_2 are in V, but neither w_3 nor w_4 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

12. (13 points) Recall that $\mathcal{P}_4$ is the vector space of polynomials of degree at most 4. Let W be the following subspace of $\mathcal{P}_4$:
$$W = \{p(x) \in \mathcal{P}_4 \mid p(1) + p(-1) = 0 \quad \text{and} \quad p(2) + p(-2) = 0\}.$$
Find a basis for W.

13. (13 points) Find an orthogonal basis for the nullspace of $A = [1 \quad 3 \quad 4 \quad 5]$. **Check your answer.**

14. (13 points) Let $A = \begin{bmatrix} 15 & -7 \\ 14 & -6 \end{bmatrix}$. Find a matrix B with $B^3 = A$. **Check your answer.**

15. (18 points) **Check your answers.** Let A be the matrix
$$A = \begin{bmatrix} 1 & 4 & 1 & 5 & 1 & 13 \\ 1 & 4 & 2 & 5 & 2 & 20 \\ 2 & 8 & 3 & 10 & 3 & 33 \end{bmatrix}$$
   (a) Find a basis for the null space of A.
   (b) Find a basis for the column space of A.
   (c) Find a basis for the row space of A.
   (d) Write each column of A as a linear combination of your answer to (b).
   (e) Write each row of A as a linear combination of your answer to (c).