Math 544, Final Exam, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 100 points. Each problem is worth 12.5 points. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences.

1. Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.

2. Define "span". Use complete sentences. Include everything that is necessary, but nothing more. (You may define "span" as a noun or as a verb; whichever you prefer.

3. Let A and B be 2×2 matrices with $A^2 = B^2$. Does B have to equal A or $-A$? If yes, prove the statement. If no, show an example.

4. Let $T\colon V \to W$ be a linear transformation of vector spaces. Let $w_1, \dots, w_b$ be a basis for the image of T; $z_1, \dots, z_a$ be a basis for the null space of T; and $v_1, \dots, v_b$ be vectors in V with $T(v_i) = w_i$ for $1 \le i \le b$. Prove that the vectors $v_1, \dots, v_b, z_1, \dots, z_a$ are linearly independent. Recall that the image of T is equal to
$$\{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

5. Suppose that A is a matrix with distinct eigenvalues λ_1 and λ_2. Suppose further that v_1 and v_2 are nonzero eigenvectors of A with v_i belonging to λ_i. Prove that v_1 and v_2 are linearly independent.

6. Let A and B be $n \times n$ matrices with A invertible. Does the column space of B have to equal the column space of AB? If yes, prove the statement. If no, give a counter example.

7. Let A and B be $n \times n$ matrices with B invertible. Does the column space of A have to equal the column space of AB? If yes, prove the statement. If no, give a counter example.

8. Find a basis for the null space of
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$