Math 544, Final Exam, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 100 points. There are **10** problems on **TWO SIDES**. Each problem is worth 10 points. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. **Make your work be coherent and clear. Write in complete sentences.**

1. Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.

2. Define "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.

3. Let $\mathcal{C}$ be the vector space of infinitely differentiable functions $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$. Consider the function T from $\mathcal{C}$ to $\mathcal{C}$ which is defined by $T(f(x)) = f''(x) + e^x \cdot f'(x) + \cos(x) \cdot f(x)$. Is T a linear transformation? If yes, **prove it**. If no, **give an example**.

4. Give a basis for the null space of $A = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 1 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

5. Give an orthogonal basis for the column space of $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 3 \\ 1 & -1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

6. Let A be the matrix $\begin{bmatrix} -3 & 10 \\ -\frac{4}{3} & \frac{13}{3} \end{bmatrix}$. Find $\lim_{n\to\infty} A^n$.

7. Let A and B be $n \times n$ symmetric matrices. **State** a necessary and sufficient condition for the matrix AB to be symmetric. **Prove** both directions of your assertion. (You are supposed to state a true fact that looks like AB is symmetric if and only if XXX. Then you are supposed to prove that if AB is symmetric, then XXX happens. Then you are supposed to prove that if XXX happens, then AB is symmetric. Of course, XXX is more interesting than merely, "AB is symmetric".)

8. Suppose that v_1, v_2, and v_3 are linearly independent vectors in $\mathbb{R}^n$ and M is an invertible $n \times n$ matrix. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.

9. Let U and V be finite dimensional subspaces of the vector space W. Recall that $U \cap V$ and $U + V$ are the vector spaces

$$U \cap V = \{w \in W \mid w \in U \text{ and } w \in V\} \quad \text{and}$$

$$U + V = \{w \in W \mid \text{ there exists } u \in U \text{ and } v \in V \text{ with } w = u + v\}.$$

Give a formula which relates the following vector space dimenions $\dim U$, $\dim V$, $\dim(U \cap V)$ and $\dim(U + V)$. Give a **complete and correct** proof of your formula.

10. Let W be the vector space of 3×3 matrices, V be the subspace of W lower triangular matrices and U be the subspace of W of upper triangular matrices. Give a basis for U, a basis for V, a basis for $U \cap V$ and a basis for $U + V$. (Recall that the matrix M from W is upper triangular if $M_{i,j} = 0$ when $j < i$ and M is *lower triangular* if $M_{i,j} = 0$ when $i < j$ for the vector spaces of upper and *lower triangular* matrices.)