**Math 544, Final Exam, Spring 2011**

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 100 points. SHOW your work. **No Calculators, Cell phones, or Computers.** Write your answers as legibly as you can. Make your work be coherent and clear. **Write in complete sentences. Check your answer**, whenever possible.

Recall that $\mathcal{P}_n$ is the vector space of polynomials of degree at most $n$. Recall that $\text{Mat}_{n\times m}(\mathbb{R})$ is the vector space of $n \times m$ matrices.

1. (5 points) **Define** "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.

2. (5 points) **Define** "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.

3. (5 points) **Define** "span". Use complete sentences. Include everything that is necessary, but nothing more.

4. (5 points) **Define** "basis". Use complete sentences. Include everything that is necessary, but nothing more.

5. (5 points) **Define** "dimension". Use complete sentences. Include everything that is necessary, but nothing more.

6. (5 points) **Define** "diagonalizable". Use complete sentences. Include everything that is necessary, but nothing more.

7. (5 points) **Define** "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.

8. (7 points) Let $T\colon V \to W$ be a linear transformation of vector spaces. Recall that the *image of* $T$ is the set

$$\text{Image}\, T = \{T(v) \mid v \in V\}.$$

Prove that the $\text{Image}\, T$ is a vector space.

**There are more problems on the other side.**

9. (7 points) Consider the linear transformation $T\colon \mathcal{P}_3 \to \mathbb{R}$ which is given by

$$T(p(x)) = \int_0^1 p(x)dx$$

for all $p(x)$ in $\mathcal{P}_3$. Find a basis for the null space of $T$.

10. (7 points) Consider the linear transformation $T\colon \text{Mat}_{2\times 2}(\mathbb{R}) \to \text{Mat}_{2\times 2}(\mathbb{R})$, which is given by

$$T(M) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} M$$

for all $M$ in $\text{Mat}_{2\times 2}(\mathbb{R})$. Find a basis for the null space of $T$.

11. (7 points) Let $M = \begin{bmatrix} 4 & \frac{21}{2} \\ -1 & \frac{-5}{2} \end{bmatrix}$. Find $\lim\limits_{n\to\infty} M^n$.

12. (8 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 12 \\ 16 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 12 \end{bmatrix}.$$

Find the matrix $M$ with $T(v) = Mv$ for all $v$ in $\mathbb{R}^2$. **CHECK your answer.**

13. (8 points) Find an orthogonal basis for the null space of $\begin{bmatrix} 1 & 3 & 4 & 5 \end{bmatrix}$. **CHECK your answer.**

14. (14 points) Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 & -3 \\ 2 & 4 & 3 & 4 & 1 & -4 \\ 2 & 4 & 2 & 2 & 1 & -5 \end{bmatrix}$. Find a basis for the null space of $A$. Find a basis for the column space of $A$. Find a basis for the row space of $A$. Express each column of $A$ in terms of your basis for the column space. Express each row of $A$ in terms of your basis for the row space. **Check your answer.**

15. (7 points) Let $T\colon V \to W$ be a linear transformation of vector spaces. Let $w_1, \dots, w_r$ be a basis for $\text{Image}\, T$. Let $v_1, \dots, v_r$ be vectors in $V$ with $T(v_i) = w_i$ for each $i$. Let $u_1, \dots, u_s$ be a basis for the null space of $T$. Prove that $u_1, \dots, u_s, v_1, \dots, v_r$ is a basis for $V$. **I expect a complete proof.** I do not expect any theorems to be quoted. “We proved this in class” is not an acceptable answer. **Write in complete sentences.**