11. (15 points) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.

(a) If A and B are 2×2 matrcies, then the null space of B is contained in the null space of AB.

(b) If A and B are 2×2 matrcies, then the null space of B is contained in the null space of BA.

Statement (a) is TRUE. Take v in the null space of B. So, $Bv = 0$. It follows that $ABv = 0$ and v is in the null space of AB.

Statement (b) is FALSE. Suppose $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Observe that $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the nullspace of B; but v is not in the null space of BA because $Bv = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $BAv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

12. (10 points) Let A be a matrix. Suppose that v_1 and v_2 are non-zero vectors and λ_1 and λ_2 are numbers with $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, and $\lambda_1 \neq \lambda_2$. PROVE that v_1 and v_2 are linearly independent.

Suppose

$$(*) \qquad c_1 v_1 + c_2 v_2 = 0$$

for some numbers c_1 and c_2. Multiply (*) by A to see that

$$c_1 A v_1 + c_2 A v_2 = 0;$$

hence

$$(**) \qquad c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0.$$

Multiply (*) by λ_1 to see

$$(***) \qquad c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0.$$

Subtract (***) from (**) to see

$$c_2(\lambda_2 - \lambda_1) v_2 = 0.$$

We know that $\lambda_2 - \lambda_1$ is a non-zero number and v_2 is a non-zero vector, so c_2 must be zero. Look at (*) to see that $c_1 v_1 = 0$, but v_1 is a non-zero vector. We conclude that c_1 and c_2 must be zero; hence, v_1 and v_2 are linearly independent.

13. (10 points) Let T be the linear transformation of $\mathbb{R}^2$ which fixes the origin, but rotates the plane in the counter clockwise direction by $\pi/4$ radians. Find the matrix M with $T(v) = Mv$ for all $v \in \mathbb{R}^2$.

$$M = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \boxed{\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}.$$