Math 544, Final Exam, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. The exam is worth a total of 100 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. (7 points) Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.

2. (7 points) Define "span". Use complete sentences. Include everything that is necessary, but nothing more.

3. (7 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.

4. (12 points) Let $A = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix}$. Find $\lim_{n\to\infty} A^n$.

5. (13 points) Let W be the subspace of $\mathbb{R}^4$ which is spanned by

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

   Find an orthogonal basis for W.

6. (13 points) Let

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} -2 \\ 20 \\ 10 \\ 33 \end{bmatrix}.$$

   (a) Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.
   (b) Find a basis for the null space of A.
   (c) Find a basis for the column space of A.
   (d) Find a basis for the row space of A.
   (e) Express each column of A in terms of your answer to (c).
   (f) Express each row of A in terms of your answer to (d).

7. (7 points) Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces. Let $w_1, \ldots, w_r$ in W be a basis for the image of T. Let $v_1, \ldots, v_r$ be vectors in V with $T(v_i) = w_i$, for $1 \le i \le r$. Let $u_1, \ldots, u_s$ in V be a basis for the null space of T. Prove that $v_1, \ldots, v_r, u_1, \ldots, u_s$ is a basis for V.

8. (7 points) Let V be a vector space and let $T\colon V \to V$ be a linear transformation. Suppose that v_1, v_2, v_3 are non-zero vectors in V which are eigenvectors which belong to three distinct eigenvalues λ_1, λ_2, λ_3, respectively. Prove that v_1, v_2, v_3 are linearly independent.

9. (7 points) Is the the determinant function from the vector space of 2×2 matrices to $\mathbb{R}$ a linear transformation? **Explain thoroughly**.

10. (7 points) Recall that $\mathbb{R}[x]$ is the vector space of polynomials in one variable x with real number coefficients. Consider the function $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which is given by $T(f) = x^2 f$ for each polynomial $f \in \mathbb{R}[x]$. Is T a linear transformation? **Explain thoroughly**.

11. (13 points) In this problem, if M is a matrix, then let $\mathcal{I}(M)$ be the Column space of M. Let A and B be $n \times n$ matrices. For each question: if the

answer is yes, then prove the statement; if the answer is no, then give a counter example.
(a) Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?
(b) Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?
(c) Suppose B is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?
(d) Suppose B is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?
(e) Suppose A is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?
(f) Suppose A is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?