Math 544, Final Exam, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. The exam is worth a total of 100 points.

SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **(7 points) Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_r$ in the vector space V are *linearly independent* if the only numbers $c_1, \dots, c_r$, with $\sum\limits_{i=1}^{r} c_i v_i = 0$, are $c_1 = \dots = c_r = 0$.

2. **(7 points) Define "span". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, v_2, \dots, v_n$ in the vector space V *span* V if every vector in V is equal to a linear combination of $v_1, v_2, \dots, v_n$.

3. **(7 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A *basis* for the vector space V is a set of vectors in V which span V and are linearly independent.

4. **(12 points) Let $A = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix}$. Find $\lim\limits_{n \to \infty} A^n$.**

You must compute the eigenvalues 1 and $\frac{1}{2}$ for A. Observe that $v_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ is an eigenvector of A which belongs to $\lambda = 1$ because

$$Av_1 = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 - 10 \\ -5 + 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = v_1.$$

Observe that $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A which belongs to $\lambda = \frac{1}{2}$ because

$$Av_2 = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+5 \\ 2-\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = \tfrac{1}{2} v_2.$$

Thus, we know $AS = SD$, where

$$S = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

It is easy to see that

$$S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

and $A = SDS^{-1}$. We see that

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} (SDS^{-1})^n = S(\lim_{n\to\infty} D^n)S^{-1} = S(\lim_{n\to\infty} \begin{bmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{bmatrix})S^{-1}$$

$$= S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -2 & -4 \end{bmatrix}$$

5. **(13 points) Let W be the subspace of $\mathbb{R}^4$ which is spanned by**

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \textbf{and} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find an orthogonal basis for W.

Let $u_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let

$$u_2' = w_2 - \frac{u_1^{\mathrm{T}} w_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Let $u_2 = 4u_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let

$$u_3' = w_3 - \frac{u_1^{\mathrm{T}} w_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} w_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Let $u_3 = 3u_3' = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$. Our orthogonal basis for W is

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}}$$

6. **(13 points) Let**

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}, \quad \textbf{and} \quad b = \begin{bmatrix} -2 \\ 20 \\ 10 \\ 33 \end{bmatrix}.$$

(a) **Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.**

We apply row operations to the augmented matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 3 & 12 & 1 & 5 & 5 & 20 \\ 2 & 8 & 1 & 3 & 2 & 10 \\ 5 & 20 & 2 & 8 & 8 & 33 \end{bmatrix}$$

Replace row 2 by row 2 minus 3 row 1.
Replace row 3 by row 3 minus 2 row 1.
Replace row 4 by row 4 minus 5 row 1 to get

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 1 & -1 & 4 & 14 \\ 0 & 0 & 2 & -2 & 13 & 43 \end{bmatrix}$$

Replace row 3 by row 3 minus row 2.
Replace row 4 by row 4 minus 2 row 2.

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 0 & 0 & -4 & -12 \\ 0 & 0 & 0 & 0 & -3 & -9 \end{bmatrix}$$

Divide row 3 by -4**. Divide row 4 by** -3**.**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Replace row 2 by row 2 minus 8 times row 3. Replace row 1 by row 1 plus row 3. Replace row 4 by row 4 minus row 3.

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution of the system of equations is

$$\boxed{\begin{array}{lccc} x_1 = & 1 & -4x_2 & -2x_4 \\ x_2 = & & x_2 & \\ x_3 = & 2 & & +x_4 \\ x_4 = & & & x_4 \\ x_5 = & 3 & & \end{array}}$$

Three specific solutions are

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 3 \end{bmatrix}.}$$

(b) Find a basis for the null space of A**.**

$$\boxed{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$$

(c) Find a basis for the column space of A.

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \quad A_{*,3} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix}}$$

(d) Find a basis for the row space of A.

$$\boxed{\begin{aligned} w_1 &= [1 \quad 4 \quad 0 \quad 2 \quad 0] \\ w_2 &= [0 \quad 0 \quad 1 \quad -1 \quad 0] \\ w_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1] \end{aligned}}$$

(e) Express each column of A in terms of your answer to (c).

$$\boxed{A_{*,2} = 4A_{*,1}, \qquad A_{*,4} = 2A_{*,1} - A*, 3.}$$

(f) Express each row of A in terms of your answer to (d).

$$\boxed{\begin{gathered} A_{1,*} = w_1 - w_3, \quad A_{2,0} = 3w_1 + w_2 + 5w_3, \\ A_{3,*} = 2w_1 + w_2 + 2w_3, \quad A_{4,*} = 5w_1 + 2w_2 + 8w_3. \end{gathered}}$$

7. **(7 points) Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces. Let $w_1, \ldots, w_r$ in W be a basis for the image of T. Let $v_1, \ldots, v_r$ be vectors in V with $T(v_i) = w_i$, for $1 \le i \le r$. Let $u_1, \ldots, u_s$ in V be a basis for the null space of T. Prove that $v_1, \ldots, v_r, u_1, \ldots, u_s$ is a basis for V.**

We first show linear independence. Suppose that a_i and b_j are numbers with

(*)
$$\sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{s} b_j u_j = 0.$$

Apply the linear transformation T to see that

$$\sum_{i=1}^{r} a_i w_i = 0.$$

The vectors $w_1, \ldots, w_r$ are linearly independent, so each a_i must be zero, Now (*) tells us that

$$\sum_{j=1}^{s} b_j u_j = 0.$$

The vectors $u_1, \ldots, u_s$ are linearly independent so each b_j must also be zero.

Now we show that $v_1, \ldots, v_r, u_1, \ldots, u_s$ span V. Take $v \in V$. The vector $T(v)$ is in the image of T, so there are numbers $a_1, \ldots, a_r$ with $T(v) = \sum\limits_{i=1}^{r} a_i w_i$. Of course $w_i = T(v_i)$. It follows that $v - \sum\limits_{i=1}^{r} a_i v_i$ is in the null space of T that is, $v - \sum\limits_{i=1}^{r} a_i v_i$ may be written as a linear combination of $u_1, \ldots, u_s$. In other words, v may be written as a linear combination of $v_1, \ldots, v_r, u_1, \ldots, u_s$, and the proof is complete.

8. **(7 points) Let V be a vector space and let $T\colon V \to V$ be a linear transformation. Suppose that v_1, v_2, v_3 are non-zero vectors in V which are eigenvectors which belong to three distinct eigenvalues λ_1, λ_2, λ_3, respectively. Prove that v_1, v_2, v_3 are linearly independent.**

We first show that v_1 and v_2 are linearly independent. Suppose a_1 and a_2 are numbers with

$$(**) \qquad a_1 v_1 + a_2 v_2 = 0.$$

Multiply (**) by λ_1 to get

$$a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2 = 0.$$

Multiply (**) by A to get

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0.$$

Subtract to learn that $a_2(\lambda_2 - \lambda_1) v_2 = 0$. The vector v_2 is not zero. The number $\lambda_2 - \lambda_1$ is not zero; so, the number a_2 must be zero. Look back at (**) to see that a_1 must also be zero.

Now we deal with all three vectors. Suppose a_1 , a_2 , and a_3 are numbers with

$$(***) \qquad a_1v_1 + a_2v_2 + a_3v_3 = 0$$

Multiply (***) by λ_3 to see

$$a_1\lambda_3v_1 + a_2\lambda_3v_2 + a_3\lambda_3v_3 = 0$$

Multiply (***) by A to see

$$a_1\lambda_1v_1 + a_2\lambda_2v_2 + a_3\lambda_3v_3 = 0$$

Subtract to learn that

$$a_1(\lambda_1 - \lambda_3)v_1 + a_2(\lambda_2 - \lambda_3)v_2 = 0.$$

We already showed that v_1 and v_2 are linearly independent; so, $a_1(\lambda_1 - \lambda_3) = 0$ and $a_2(\lambda_2 - \lambda_3) = 0$. The numbers $\lambda_2 - \lambda_3$ and $\lambda_1 - \lambda_3$ are not zero; hence a_1 and a_2 must be zero. Look back at (***) to see that a_3 must also be zero.

9. **(7 points) Is the the determinant function from the vector space of 2×2 matrices to $\mathbb{R}$ a linear transformation? Explain thoroughly.**

NO. The determinant of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ plus the determinant of $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is zero plus zero which is zero. On the other hand if I add the matrices first and then take the determinant I get the determinant of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is 1 .

10. **(7 points) Recall that $\mathbb{R}[x]$ is the vector space of polynomials in one variable x with real number coefficients. Consider the function $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which is given by $T(f) = x^2f$ for each polynomial $f \in \mathbb{R}[x]$. Is T a linear transformation? Explain thoroughly.**

Yes. If f and g are polynomials, then

$$T(f+g) = x^2(f+g) = x^2f + x^2g = T(f + T(g).$$

If c is a number, then

$$T(cf) = x^2cf = cx^2f = cT(f).$$

11. **(13 points) In this problem, if M is a matrix, then let $\mathcal{I}(M)$ be the Column space of M. Let A and B be $n \times n$ matrices. For each question: if the answer is yes, then prove the statement; if the answer is no, then give a counter example.**

(a) **Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take B to be the identity matrix and A to be the zero matrix.

(b) **Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the column space of AB and not in the column space of B.

(c) **Suppose B is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take B to be the identity matrix and A to be the zero matrix.

(d) **Suppose B is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

Yes. In this case the column space of B is all of $\mathbb{R}^n$.

(e) **Suppose A is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the column space of B and not in the column space of AB.

(f) **Suppose A is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the column space of AB and not in the column space of B.