Math 544, Exam 4, Fall 2006, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. The exam is worth a total of 50 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Define "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.**

The function T from the vector space V to the vector space W is a *linear transformation* if $T(v_1+v_2)=T(v_1)+T(v_2)$ and $T(rv_1)=rT(v_1)$ for all v_1 and v_2 in V and all r in $\mathbb{R}$.

2. **Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The dimension of the vector space V is the number of vectors in a basis for V.

3. **Find all eigenvalues and eigenvectors of the matrix $A=\begin{bmatrix}-1 & -6\\ 1 & 4\end{bmatrix}$. CHECK your answer.**

To find the eigenvalues of A, we solve $0=\det(A-\lambda I)=\begin{vmatrix}-1-\lambda & -6\\ 1 & 4-\lambda\end{vmatrix}=$ $(-1-\lambda)(4-\lambda)+6=\lambda^2-3\lambda+2=(\lambda-2)(\lambda-1)$. We conclude that the eigenvalues of A are 1 and 2.

The eigenvectors that belong to $\lambda = 1$ are the vectors in the nullspace of $A - I = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$. Exchange row 1 and row 2: $\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$. Replace row 2 with row 2 plus 2 times row 1: $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$. The nullspace of $A - I$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 = -3x_2$ and x_2 is arbitrary. So,

$$\boxed{v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \text{ is a basis for the eigenspace of } A \text{ belonging to the eigenvector } \lambda = 1.}$$

The eigenvectors that belong to $\lambda = 2$ are the vectors in the nullspace of $A - 2I = \begin{bmatrix} -3 & -6 \\ 1 & 2 \end{bmatrix}$. Exchange row 1 and row 2: $\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$. Replace row 2 with row 2 plus 3 times row 1: $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The nullspace of $A - 2I$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 = -2x_2$ and x_2 is arbitrary. So,

$$\boxed{v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ is a basis for the eigenspace of } A \text{ belonging to the eigenvector } \lambda = 2.}$$

Check: We see that $Av_1 = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-6 \\ -3+4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} = 1v_1 \checkmark$.
We also see that $Av_2 = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-6 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2v_2. \checkmark$

4. **Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \sqrt{3}x$. Find the matrix M with $T(v) = Mv$ for all $v \in \mathbb{R}^2$.**

The line $y = \sqrt{3}\,x$ makes the angle $\theta = \frac{\pi}{3}$ with the x-axis. (If need be draw the right triangle with base 1 and height $\sqrt{3}$. The hypotenuse is 2. So the angle of inclination, θ, has $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$ and $\sin\theta = \frac{\text{op}}{\text{hyp}} = \frac{\sqrt{3}}{2}$. Thus $\theta = \frac{\pi}{3}$.) It follows that

$$M = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}.$$

The best check is to make sure that $Mv = v$ for some vector on $y = \sqrt{3}\,x$ (like for example $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$); and $Mw = -w$ for some vector perpendicular to $y = \sqrt{3}\,x$ (like for example $w = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$). This happens.

5. **Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the matrix M with $T(v) = Mv$ for all $v \in \mathbb{R}^2$.**

I would like to know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. It is easy to see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The function T is a linear transformation, so, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$. In a similar manner, we see that

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

It follows that

$$M = \begin{bmatrix} -4 & 7 \\ -2 & 6 \end{bmatrix}.$$

We check that

$$M\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

and

$$M \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

6. **Give an example of a NON-ZERO 2×2 matrix A whose only eigenvalue is zero.**

We loook for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with at least one of the numbers a, b, c, or d, not equal to zero, but $\det(A - \lambda I) = \lambda^2$. Keep in mind that λ^2 is the only polynomial of degree two whose only root is $\lambda = 0$. We want $\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2$. We want $\lambda^2 + (-a - d)\lambda + (ad - bc)$ to be λ^2. One way to do this is to take $a = d = c = 0$ and $b = 1$. So

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is a non-zero matrix whose only eigenvalue is } \lambda = 0.}$$

7. **Solve $Ax = b$ for**

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

You might want to notice that the columns of A form an orthogonal set. CHECK your answer.

I will take advantage of the fact that the columns of A form an orthogonal set. If x is a vector with $Ax = b$, then $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$. That is,

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x = \begin{bmatrix} 10 \\ -2 \\ -4 \\ 0 \end{bmatrix}.$$

In other words, $x = \frac{1}{4} \begin{bmatrix} 10 \\ -2 \\ -4 \\ 0 \end{bmatrix} = \boxed{\frac{1}{2} \begin{bmatrix} 5 \\ -1 \\ -2 \\ 0 \end{bmatrix}}.$

Check We see that $A\frac{1}{2}\begin{bmatrix}5\\-1\\-2\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\4\\6\\8\end{bmatrix}$. ✓

8. **Suppose that the non-zero vectors v_1, v_2, v_3 form an orthogonal set. Prove that v_1, v_2, v_3 are linearly independent. Give a complete proof.**

Suppose c_1, c_2, and c_3 are numbers with

(*) $$c_1v_1 + c_2v_2 + c_3v_3 = 0.$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}}v_1 + c_2 \cdot v_1^{\mathrm{T}}v_2 + c_3 \cdot v_1^{\mathrm{T}}v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}}v_2 = 0$ and $v_1^{\mathrm{T}}v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}}v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}}v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (*) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}}v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}}v_2 \neq 0$. Multiply (*) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.

9. **Suppose $V_1 \subseteq V_2 \subseteq V_3$ are vector spaces and v_1, v_2, v_3, v_4 are vectors in V_3 which form a basis for V_3. Suppose further, that v_1, v_2, v_3 are in V_2 and $v_4 \notin V_2$. Suppose v_1, v_2 are in V_1 and $v_3 \notin V_1$. Do you have enough information to know the dimension of V_1. Explain very thoroghly.**

You proved on the last exam that if $U \subseteq W$ are finite dimensional vector spaces with $U \neq W$, then $\dim U < \dim W$. We will use this fact twice in the present problem. We will also use the fact that if r linearly independent vectors live in a vector space U, then $\dim U \geq r$.

The vector space V_3 has dimension 4 because it has a basis with four vectors. The vector space V_2 is a proper subspace of V_3 because v_4 is in V_3, but not in V_2. It follows that the dimension of V_2 must be less than 4. On the other hand, the vectors v_1, v_2, v_3 are linearly independent vectors in V_2; so $\dim V_2 \geq 3$. We have shown that $\dim V_2$ must equal 3. The vector space V_1 is a proper subspace of V_2; hence $\dim V_1 \leq 2$. We have exhibited 2 linearly independent vectors in V_2; thus, $\dim V_2 \geq 2$; and in fact, $\boxed{\dim V_1 \text{ must equal } 2}$.

10. **Find an orthogonal basis for the null space of** $A = [1 \quad 2 \quad 3 \quad 5]$**. CHECK your answer.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the Gram-Schmidt orthogonalization process to this basis. Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

(Notice that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$.) Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\frac{15}{70}}_{\frac{3}{14}} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}$$

Let

$$u_3 = \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}.$$

It is easy to check that

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}}$$

is an orthogonal basis for the null space of A.