Math 544, Exam 1, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points. There are 10 problems. Each problem is worth 5 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime Wednesday afternoon.

I will grade the exam Wednesday afternoon. If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Find the GENERAL solution of the system of linear equations** $Ax = b$**. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}.$$

Start with the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 1 & 6 & 3 \\ 1 & 2 & 3 & 4 & 2 & 12 & 5 \\ 2 & 4 & 6 & 8 & 3 & 18 & 8 \end{array}\right].$$

Replace row 2 with row 2 minus row 1.

Replace row 3 with row 3 minus 2 times row 1.

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 1 & 6 & 2 \end{array}\right].$$

Replace row 1 with row 1 minus row 2.
Replace row 3 with row 3 minus row 2.

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution set is

$$\boxed{\begin{array}{ccccccc} x_1 & = & 1 & -2x_2 & -3x_3 & -4x_4 & \\ x_2 & = & & x_2 & & & \\ x_3 & = & & & x_3 & & \\ x_4 & = & & & & x_4 & \\ x_5 & = & 2 & & & & -6x_6 \\ x_6 & = & & & & & x_6 \\ \multicolumn{7}{c}{\text{where } x_2,\ x_3,\ x_4, \text{ and } x_6 \text{ are arbitrary.}} \end{array}}$$

Some particular solutions are

$$v_1 =, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

These vectors were obtained by setting $x_2 = x_3 = x_4 = x_6 = 0$ (for v_1); $x_2 = 1$, $x_3 = x_4 = x_6 = 0$ (for v_2); $x_2 = 0$, $x_3 = 1$, $x_4 = x_6 = 0$ (for v_3); $x_2 = x_3 = 0$, $x_4 = 1$, $x_6 = 0$ (for v_4); and $x_2 = x_3 = x_4 = 0$, $x_6 = 1$ (for v_5). We check these specific solutions:

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+5 \\ 2+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+2 \\ -1+2+4 \\ -2+4+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+3+2 \\ -2+3+4 \\ -4+6+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+4+2 \\ -3+4+4 \\ -6+8+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

$$Av_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-4+6 \\ 1-8+12 \\ 2-12+18 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

2. **Consider the system of linear equations.**

$$\begin{aligned} x_1 + (a-1)x_2 &= 4 \\ ax_1 + 6x_2 &= 12. \end{aligned}$$

(a) **Which values for a cause the system to have no solution?**
(b) **Which values for a cause the system to have exactly one solution?**
(c) **Which values for a cause the system to have an infinite number of solutions?**

Explain thoroughly.

Consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ a & 6 & 12 \end{array}\right].$$

Replace row 2 by row 2 minus a times row 1 to get:

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & 6-a(a-1) & 12-4a \end{array}\right].$$

Multiply row 2 by -1 to get

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & a^2-a-6 & 4a-12 \end{array}\right],$$

which is the same as

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & (a-3)(a+2) & 4(a-3) \end{array}\right].$$

We see that if a is any number other than 3 or -2, then the system of equations has a unique solution. (In this case the bottom row tells the value of x_2 and the top row tells the value of x_1.) If $a = 3$, then the bottom row is completely zero and the solution set is the line described by the top equation. If $a = -2$, then the system of equations has no solution because 0 is never equal to -20. To summarize:

(a) If $a = -2$, then the system of equations has no solution.
(b) If $a \neq -2$ and $a \neq 3$, then the system of equations has exactly one solution.
(b) If $a = 3$, then the system of equations has an infinite number of solutions.

3. **Are the vectors**

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

linearly independent? Explain thoroughly.

These vectors are DEPENDENT because $-2v_1 + v_2 + v_3 = 0$. We have exhibited a non-trivial linear combination of the vectors which equals the zero vector.

4. **Suppose v_1 , v_2 and v_3 are vectors in $\mathbb{R}^3$ with v_1, v_2 linearly independent, v_1, v_3 linearly independent, and v_2, v_3 linearly independent. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, give a proof. If no, give an example.**

NO. The vectors of problem 3 give an example. The vectors v_1 and v_2 are independent (since neither vector is a multiple of the other). The vectors v_1 and v_3 are independent (since neither vector is a multiple of the other). The vectors v_2 and v_3 are independent (since neither vector is a multiple of the other). Yet the vectors v_1, v_2, v_3 are dependent because $-2v_1 + v_2 + v_3 = 0$.

5. **How many solutions does a homogeneous system of 3 linear equations in 4 unknowns have? Justify your answer very thoroughly.**

This system of equations MUST have an infinite number of solutions. It is guaranteed at least one solution because it is a homogeneous system of equations. Furthermore, there are more variables than equations so some column in the reduced matrix does not contain a leading one. This column corresponds to a variable which is free to take on any value.

6. **How many solutions does a homogeneous system of 4 linear equations in 3 unknowns have? Justify your answer very thoroughly.**

This system of equations MIGHT have a unique solution or MIGHT have an infinite number of solutions. The system of equations is homogeneous, so the system has at least one solution. At any rate, the reduced matrix will have at most 3 leading ones. If there are exactly 3 leading ones, then the system of equations has a uniques solution. If there are fewer than 3 leading ones (In other words, if there were "really" only two equations and the other two equations can be expressed as linear combinations of the first two equations.), then the system of equations would have an infinite number of solutions.

7. **Recall that the matrix A is *symmetric* if $A^{\mathrm{T}} = A$. Let A and B be 2×2 symmetric matrices. Give an example to show that AB does not have to be a symmetric matrix.**

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that A and B are each symmetric, but the product

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not symmetric.

8. **Give a condition (*) so that if A and B are 2×2 symmetric matrices which satisfy (*), then AB also is a symmetric matrix.**

If $\boxed{AB = BA}$, then AB is also a symmetric matrix because

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = BA = AB.$$

The first equality holds for all matrices. The send equality holds because B and A are symmtric. The last equality is our new hypothesis.

9. **List four different 2×2 matrices X which satisfy $X^2 - 2X = 0$.**

The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ \frac{1}{b} & 1 \end{bmatrix}$$

all work for any number $b \neq 0$.

10. **Find a matrix** B **with** $AB = C$ **for** $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ **and** $C = \begin{bmatrix} 2 & 6 \\ 3 & 6 \end{bmatrix}$**.**

Let $B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$. We solve $A \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and we solve $A \begin{bmatrix} b_{1,2} \\ b_{2,2} \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. We solve the two systems of equations simultaneously. That is, we consider the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 2 & 6 \\ 1 & 4 & 3 & 6 \end{array}\right].$$

When we put the matrix in Reduced Row Echelon Form, the left most column of the augmentation will tell us about $\begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}$ and the right most column of the augmentaion will tell us about $\begin{bmatrix} b_{1,2} \\ b_{2,2} \end{bmatrix}$. Replace row 2 by row 2 minus row 1 to get:

$$\left[\begin{array}{cc|cc} 1 & 3 & 2 & 6 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Replace row 1 by row 1 minus 3 times row 2 to get:

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 6 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

We conclude that

$$\boxed{B = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}}$$

This is correct because

$$AB = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1+3 & 6+0 \\ -1+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 3 & 6 \end{bmatrix} = C \checkmark$$