Math 544, Final Exam, Fall 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 16 problems. Problem 1 is worth 10 points. Each of the other problems is worth 6 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I WILL GRADE YOUR EXAM ON FRIDAY. Once your exam is graded, I will send your grade to VIP. If the grade isn't on VIP, then I also do not know it. If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly after the exam is finished.

1. Let
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

   (a) Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.
   (b) Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.
   (c) Find a basis for the null space of A.
   (d) Find a basis for the column space of A.
   (e) Find a basis for the row space of A.
   (f) Express each column of A in terms of your answer to (d).
   (g) Express each row of A in terms of your answer to (e).

2. State any two of the four dimension Theorems.

3. Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.

4. Define "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.

5. Define "diagonalizable". Use complete sentences. Include everything that is necessary, but nothing more.

6. Define "nonsingular". Use complete sentences. Include everything that is necessary, but nothing more.

7. Let A be an $n \times n$ matrix. Record eight statements that are equivalent to "the matrix A is invertible".

8. Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree less than or equal to three. Let $T\colon \mathcal{P}_3 \to \mathbb{R}$ be the linear transformation which is given by $T(p(x)) = \int_{-1}^{1} p(x)dx$. Find a basis for the null space of T.

9. Let A be a square matrix, v_1 and v_2 be non-zero vectors with $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where λ_1 and λ_2 are real numbers with $\lambda_1 \neq \lambda_2$. Prove that $\{v_1, v_2\}$ is a linearly independent set of vectors.

10. Find an orthogonal basis for the null space of $A = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}$.

11. Let $A = \begin{bmatrix} 5 & -2 \\ \frac{28}{3} & -\frac{11}{3} \end{bmatrix}$. Find $\lim\limits_{n\to\infty} A^n$.

12. Consider the function $T\colon \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ which sends a 2×2 matrix A to the real number $\det(A)$. Is T a linear transfromation? Explain.

13. Express $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$. (You are welcome to notice that u_1, u_2, u_3 form an orthogonal set of vectors.) **Check your answer.**

14. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \notin V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V? Explain **very thoroughly**.

15. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \in V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V? Explain **very thoroughly**.

16. Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Prove **very thoroughly** that v_1, v_2, and v_3 are linearly independent.