Math 544, Exam 4, Fall 2005 Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you. There are 10 problems. Each problem is worth 5 points. The exam worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly after the exam is finished.

Recall that the matrix A is *symmetric* if $A^{\mathrm{T}} = A$ and the matrix A is *skew-symmetric* if $A^{\mathrm{T}} = -A$.

1. **Let A be a 2×2 symmetric matrix with real number entries. Does A HAVE to have real eigenvalues? If yes, PROVE it. If no, then give a counter example.**

$\boxed{\text{YES}}$ Write $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. The eigenvalues of A are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - b^2$$

$$= \lambda^2 + (-a-d)\lambda + ad - b^2.$$

The quadratic formula tells us that the solutions of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0$$

are

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

In our problem, $A = 1$, $B = -a - d$ and $C = ad - b^2$. The eigenvalues of A are

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2}$$

$$= \frac{a + d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2}$$

$$= \frac{a + d \pm \sqrt{a^2 - 2ad + d^2 + 4b^2}}{2}$$

$$= \frac{a + d \pm \sqrt{(a-d)^2 + 4b^2}}{2}.$$

Notice that the number under the radical is the SUM of two perfect squares. This number is ZERO or HIGHER. So, when we take the square root, we will get real number answers (rather than complex numbers which are not real).

2. **Let A be a 2×2 skew-symmetric matrix with real number entries. Does A HAVE to have real eigenvalues? If yes, PROVE it. If no, then give a counter example.**

$\boxed{\text{NO}}$ The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a 2×2 skew-symmetric matrix with real number entries, but the eigenvalues of A are i and $-i$, which are complex, but not real. Notice that $\det(A - \lambda I) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$.

3. **Let A be a 3×3 skew-symmetric matrix with real number entries. Does A have to be singular? If yes, PROVE it. If no, then give a counter example.**

$\boxed{\text{YES}}$ We know that

$$\det A = \det(A^{\mathrm{T}}) = \det(-A) = (-1)^3 \det A = -\det A.$$

The first equality holds for all matrices. The second equality holds because A is skew-symmetric. The third equality holds because we had to pull minus one out from each of three rows.

Add $\det A$ to both sides to get $2 \det A = 0$. Divide by two to see $\det A = 0$. Conclude that A is singular.

4. **Let A be a 2×2 skew-symmetric matrix with real number entries. Does A have to be singular? If yes, PROVE it. If no, then give a counter example.**

$\boxed{\text{NO}}$ The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a 2×2 skew-symmetric matrix with real number entries but A is non-singular because $\det A = 1 \neq 0$.

5. **Find the inverse of**

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}.$$

It might be worth your while to notice that the columns of A form an orthogonal set.

The columns of A form an orthogonal set, so A^{T} is fairly close to A^{-1}. Indeed,

$$A^{\mathrm{T}} A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Multiply both sides of the above equation by

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

to see that

$$\left(\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} A^{\mathrm{T}} \right) A = I.$$

It follows that

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} A^{\mathrm{T}} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}}.$$

6. **Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}6\\7\end{bmatrix}$. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.**

We see that $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right)$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right)$. The function T is a linear transformation; hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}6\\7\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix}4\\5\end{bmatrix} - \begin{bmatrix}6\\7\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\end{bmatrix}$$

and

$$\boxed{M = \begin{bmatrix}5 & -1\\6 & -1\end{bmatrix}.}$$

7. **Let V be the set of 2×2 singular matrices. Is V a vector space? Explain thoroughly.**

$\boxed{\text{NO}}$. The set V is not closed under addition. Indeed, the matrices

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

are both in V, but their sum, which is the identity matrix, is not in V.

8. **Let V be the set of 2×2 matrices with trace 0. Is V a vector space? Explain thoroughly. Recall that the *trace* of a square matrix is the sum of the elements on its main diagonal.**

$\boxed{\text{YES}}$. We see that the zero matrix is in V. We also see that V is closed under addition. Indeed, if

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}a' & b'\\c' & d'\end{bmatrix}$$

are both in V , then $a+d=0$ and $a'+d'=0$. It follows that the sum of the two matrices, which is equal to

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix},$$

is also in V , because $(a+a')+(d+d')=(a+d)+(a'+d')=0$. Finally, V is closed under scalar multiplication. If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is in V and r is a number, then $a+d=0$ and

$$r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

is in V because $ra+rd=r(a+d)=0$.

9. **Find a matrix B with $B^2 = A$ for $A = \begin{bmatrix} 34 & 75 \\ -10 & -21 \end{bmatrix}$. Check your answer.**

We see that

$$\det(A-\lambda I) = (34-\lambda)(-21-\lambda)+750 = \lambda^2 - 13\lambda + 36 = (\lambda-4)(\lambda-9).$$

We see that

$$A\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 9\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Let $S = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. We have just calculated that $AS = SD$. We take

$$B = S\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1} = \begin{bmatrix} 10 & 9 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & 15 \\ -2 & -3 \end{bmatrix}}.$$

Check: We see that

$$B^2 = \begin{bmatrix} 8 & 15 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 8 & 15 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 64-30 & 120-45 \\ -16+6 & -30+9 \end{bmatrix} = \begin{bmatrix} 34 & 75 \\ -10 & -21 \end{bmatrix}. \checkmark$$

10. **Let** A **be a square matrix,** v_1 **and** v_2 **be non-zero vectors with** $Av_1 = \lambda_1 v_1$ **and** $Av_2 = \lambda_2 v_2$ **, where** λ_1 **and** λ_2 **are real numbers with** $\lambda_1 \neq \lambda_2$ **. Prove that** $\{v_1, v_2\}$ **is a linearly independent set of vectors.**

Suppose

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiply both sides of (1) by A to get

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0. \tag{2}$$

Multiply both sides of equation (1) by λ_2 to get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0. \tag{3}$$

Subtract (2) minus (3) to get

$$c_1(\lambda_1 - \lambda_2) v_1 = 0.$$

The vector v_1 is not zero. If a scalar times v_1 is zero, then the scalar must be zero. Thus, the scalar $c_1(\lambda_1 - \lambda_2) = 0$. But, $(\lambda_1 - \lambda_2)$ is not zero; so, c_1 must be zero. Equation (1) now says that $c_2 v_2 = 0$. The vector v_2 is not zero; so, the scalar c_2 must be zero.