Math 544, Exam 3, Fall 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you. There are 10 problems. The exam worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly after the exam is finished.

1. (8 points) **Let**

$$A = \begin{bmatrix} 2 & 6 & 2 & 8 & 2 \\ 2 & 6 & 3 & 11 & 2 \\ 4 & 12 & 5 & 19 & 5 \\ 2 & 6 & 2 & 8 & 2 \end{bmatrix}.$$

   (a) Find a basis for the null space of A**.**
   (b) Find a basis for the column space of A**.**
   (c) Find a basis for the row space of A**.**
   (d) Write each column of A **as a linear combination of your answer to (b).**
   (e) Write each row of A **as a linear combination of your answer to (c).**

Replace $R1$ by $\frac{1}{2}R1$ to get

$$\begin{bmatrix} 1 & 3 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 2 \\ 4 & 12 & 5 & 19 & 5 \\ 2 & 6 & 2 & 8 & 2 \end{bmatrix}$$

Apply $R2 \mapsto R2 - 2R1$, $R3 \mapsto R3 - 4R1$ and $R4 \mapsto R4 - 2R1$ to get

$$\begin{bmatrix} 1 & 3 & 1 & 4 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply $R1 \mapsto R1 - R2$ and $R3 \mapsto R3 - R2$ to get

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply $R1 \mapsto R1 - R3$ to get

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors in the null space of A have the form

$$\begin{aligned} x_1 &= -3x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= -3x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

So the vectors

$$\boxed{v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \quad (a)}$$

are a basis for the null space of A. (Do check that $Av_1 = 0$ and $Av_2 = 0$.) Columns 1, 3, and 5 in the reduced matrix have leading ones. So columns 1, 3, and 5 from the original matrix form a basis for the column space of the original matrix. In other words,

$$\boxed{A_{*,1} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \quad A_{*,3} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \text{and} \quad A_{*,5} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 2 \end{bmatrix} \quad (b)}$$

form a basis for the column space of A. The fact that v_1 is in the null space of A tells me that $A_{*,2} = 3A_{*,1}$ and the fact that v_2 is in the null space of A tells me that $A_{*,4} = A_{*,1} + 3A_{*,3}$. Thus,

$$\boxed{\begin{aligned} A_{*,1} &= A_{*,1} \\ A_{*,2} &= 3A_{*,1} \\ A_{*,3} &= A_{*,3} \\ A_{*,4} &= A_{*,1} + 3A_{*,3} \\ A_{*,5} &= A_{*,5} \end{aligned} \quad (d)}.$$

The non-zero rows of the reduced matrix are a basis for the row space of the original matrix; that is

$$\boxed{\begin{array}{l} w_1 = [1 \quad 3 \quad 0 \quad 1 \quad 0], \\ w_2 = [0 \quad 0 \quad 1 \quad 3 \quad 0], \\ w_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1] \end{array} \quad (c)}$$

is a basis for the row space of A. It is clear that

$$\boxed{\begin{array}{l} A_{1,*} = 2w_1 + 2w_2 + 2w_3 \\ A_{2,*} = 2w_1 + 3w_2 + 2w_3 \\ A_{3,*} = 4w_1 + 5w_2 + 5w_3 \\ A_{4,*} = 2w_1 + 2w_2 + 2w_3 \end{array} \quad (e)}$$

2. (6 points) **Find an orthogonal basis for the vector space spanned by**

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let $u_1 = v_1$. Let

$$u_2 = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{5}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}.$$

Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{25} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 10 \\ -5 \\ 4 \\ -3 \end{bmatrix}.$$

Let $u_3 = 25u_3'$. So, our answer is:

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 10 \\ -5 \\ 4 \\ -3 \end{bmatrix}.}$$

We check that u_1, u_2, and u_3 form an orthogonal set, and that u_1, u_2, u_3 span the same vector space as v_1, v_2, v_3. Indeed, we notice that $v_1 = u_1$, $v_2 = u_1 + u_2$, and $v_3 = \frac{3}{5}u_1 + \frac{7}{25}u_2 + \frac{1}{25}u_3$.

3. (4 points) **Express** $v = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ **as a linear combination of**

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \ \textbf{and} v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

You are encouraged to notice that v_1, v_2, v_3 **is an orthogonal set.**

Suppose $v = c_1v_1 + c_2v_2 + c_3v_3$. Multiply by v_i^{T} to see that $4 = 4c_1$, $-2 = 4c_2$, and $6 = 4c_3$. So $\boxed{c_1 = 1, \ c_2 = -\frac{1}{2}, \ c_3 = \frac{3}{2}}$. We check

$$v_1 - \frac{1}{2}v_2 + \frac{3}{2}v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} = v. \checkmark$$

4. (8 points) **Let** A **be a non-singular** $n \times n$ **matrix and** B **be an** $n \times n$ **matrix? Answer each question. If the answer is "yes", prove the statement. If the answer is "no", give an example.**
   (a) Does the column space of AB **have to equal the column space of** B**?**
   (b) Does the null space of AB **have to equal the null space of** B**?**
   (a) Does the rank space of AB **have to equal the rank space of** B**?**

   (a) $\boxed{\text{no}}$ If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; and therefore B and AB have different column spaces. (The column space of B is the "x-axis". The column space of AB is the "y-axis".)

   (b) $\boxed{\text{yes}}$ If v is in the null space of B, then $Bv = 0$; so $ABv = 0$ and v is in the null space of AB. If v is in the null space of AB, then $ABv = 0$; however A is non-singular, so Bv also has to be zero; hence, v is in the null space of B.

   (c) $\boxed{\text{yes}}$ Part (b) proves that B and AB have the same nullity. These matrices also have the same number of columns. The rank-nullity Theorem (see theorem 4 of number 6) now guarantees that B and AB have the same rank.

5. (4 points) **Let**

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \notin V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V? Explain very thoroughly.

$\boxed{\text{NO}}$. The vector space V could have dimension 2. (In this case v_1 and v_2 are a basis for V.) On the other hand, the vector space V could have dimension 3. For example, the vector space V spanned by v_1, v_2, and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

has dimension 3 and does not contain v_3 or v_4.

6. (4 points) **State any one of the four dimension Theorems.**

<u>Theorem 1.</u> If V is a subsapce of $\mathbb{R}^n$, then every basis for V has the same number of vectors.

<u>Theorem 2.</u> If V is a subsapce of $\mathbb{R}^n$, then every linearly independent subset in V is part of a basis for V.

<u>Theorem 3.</u> If V is a subsapce of $\mathbb{R}^n$, then every finite spanning set for V contains a basis for V.

<u>Theorem 4.</u> If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A.

7. (4 points) **Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A <u>basis</u> for a vector space V is a linearly independent subset of V which spans V.

8. (4 points) **Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The <u>dimension</u> of a vector space V is the number of vectors in a basis for V.

9. (4 points)**Let** $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{array}{l} x_1 + 3x_2 + 4x_3 = 1 \\ 2x_1 + 9x_2 + 5x_3 = 0 \\ 5x_1 + 14x_2 + 41x_3 = 0 \\ -x_1 + 32x_2 + 12x_3 = 0 \end{array} \right\}$**. Is** V **a vector space? Explain thoroughly.**

NO. The zero vector is not in V.

10. (4 points)**Let** $V = \left\{ \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 9x_2 + 5x_3 \\ 5x_1 + 14x_2 + 41x_3 \\ -x_1 + 32x_2 + 12x_3 \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}$**. Is** V **a vector space? Explain thoroughly.**

YES. The set V is the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 5 \\ 5 & 14 & 41 \\ -1 & 32 & 12 \end{bmatrix}.$$

We proved that the column space of every matrix is a vector space.