Math 544, Exam 2, Fall 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you. There are 10 problems. Each problem is worth 5 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly after the exam is finished.

1. **Find the GENERAL solution of the system of linear equations** $Ax = b$**. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & 2 & 10 \\ 1 & 2 & 3 & 13 \\ 2 & 4 & 5 & 23 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}.$$

We apply Gaussian Elimination to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 10 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ 2 & 4 & 5 & 23 & -3 \end{array}\right].$$

Replace $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 - 2R1$ to get

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 10 & -1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right].$$

Replace $R1 \mapsto R1 - 2R2$ and $R3 \mapsto R3 - R2$ to get

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$\boxed{\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \quad \begin{matrix} \text{for some numbers} \\ x_2 \text{ and } x_4 \end{matrix} \right\}.}$$

Specific Solution 1. Take $x_2 = x_4 = 0$. We see that $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is a solution of $Ax = b$ because

$$Av_1 = A = \begin{bmatrix} 1 & 2 & 2 & 10 \\ 1 & 2 & 3 & 13 \\ 2 & 4 & 5 & 23 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 1-3 \\ 2-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = b. \checkmark$$

Specific Solution 2. Take $x_2 = 1$ and $x_4 = 0$. We see that $v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ is a solution of $Ax = b$ because

$$Av_2 = A = \begin{bmatrix} 1 & 2 & 2 & 10 \\ 1 & 2 & 3 & 13 \\ 2 & 4 & 5 & 23 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2-2 \\ -1+2-3 \\ -2+4-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = b. \checkmark$$

Specific Solution 3. Take $x_2 = 0$ and $x_4 = 1$. We see that $v_3 = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ is a solution of $Ax = b$ because

$$Av_3 = A = \begin{bmatrix} 1 & 2 & 2 & 10 \\ 1 & 2 & 3 & 13 \\ 2 & 4 & 5 & 23 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3-8+10 \\ -3-12+13 \\ -6-20+23 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = b. \checkmark$$

2. **Let U and V be subspaces of $\mathbb{R}^n$. Does the union $U \cup V$ have to be a subspace of $\mathbb{R}^n$? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{NO.}}$ Let $n = 2$, $U = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$, and $V = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$. We see that U and V are subspaces of $\mathbb{R}^2$; but the union $U \cup V$ is not a subspace of $\mathbb{R}^2$ because this union is not closed under addition. In particular, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in U (hence also $U \cup V$) and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in V (hence also in $U \cup V$); but the sum $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in either U or V; so $u + v \notin U \cup V$.

3. **Let** $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{array}{l} x_1 + 3x_2 + 4x_3 = 0 \\ 2x_1 + 9x_2 + 5x_3 = 0 \\ 5x_1 + 14x_2 + 41x_3 = 0 \\ -x_1 + 32x_2 + 12x_3 = 0 \end{array} \right\}$. **Is V a vector space? Explain thoroughly.**

$\boxed{\text{Yes.}}$ The set V is the null space of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 5 \\ 5 & 14 & 41 \\ -1 & 32 & 12 \end{bmatrix}.$$

We proved that the null space of every matrix is a vector space.

4. **Let** $V = \left\{ \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 9x_2 + 5x_3 \\ 5x_1 + 14x_2 + 41x_3 \\ -x_1 + 32x_2 + 12x_3 \end{bmatrix} \in \mathbb{R}^4 \middle| \, x_1, x_2, x_3 \in \mathbb{R} \right\}$. **Is V a vector space? Explain thoroughly.**

$\boxed{\text{Yes.}}$ The set V is the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 5 \\ 5 & 14 & 41 \\ -1 & 32 & 12 \end{bmatrix}.$$

We proved that the column space of every matrix is a vector space.

5. **Let** $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x_1x_2 = 0 \right\}$**. Is** V **a vector space? Explain thoroughly.**

$\boxed{\text{NO.}}$ The set V is not closed under addition. We see that $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in V; but $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not V.

6. **Define "null space". Use complete sentences. Include everything that is necessary, but nothing more.**

The <u>null space</u> of the matrix A is the set of all column vectors x with $Ax = 0$.

7.

(a) Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.

The square matrix A is <u>non-singular</u> if the **only** column vector x with $Ax = 0$ is $x = 0$.

(b) Let A **be an** $n \times n$ **matrix. List three statements that are equivalent to the statement "** A **is non-singular".**

1. The columns of A are linearly independent.
2. For every vector b in $\mathbb{R}^n$, the system of equations $Ax = b$ has a unique solution.
3. The matrix A is invertible.

8. **Let** A **and** B **be** 2×2 **matrices with** A **not equal to the zero matrix and** $A^2 = AB$**. Does** A **have to equal** B**? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{NO.}}$ Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 75 & 82 \\ 0 & 0 \end{bmatrix}$. We see that A is not the zero matrix and A does not equal B; but, A^2 and AB are both equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

9. **Let** A **and** B **be** $n \times n$ **matrices. At least one of the following statements is always true. Pick a true statement and prove it.**
   (a) The column space of A **is a subset of the column space of** AB**.**
   (b) The column space of B **is a subset of the column space of** AB**.**

(c) The column space of AB is a subset of the column space of A.
(d) The column space of AB is a subset of the column space of B.

$\boxed{\text{Statement (c) is true.}}$ If v is in the column space of AB, then $v = ABw$ for some vector w. Thus, $v = A(Bw)$ for the vector Bw and v is in the column space of A.

10. **Let A and B be $n \times n$ matrices. At least one of the following statements is always true. Pick a true statement and prove it.**
(a) The null space of A is a subset of the null space of AB.
(b) The null space of B is a subset of the null space of AB.
(c) The null space of AB is a subset of the null space of A.
(d) The null space of AB is a subset of the null space of B.

$\boxed{\text{Statement (b) is true.}}$ If v is in the null space of B, then $Bv = 0$. Multiply both sides of the equation on the left by A to see that $ABv = 0$. We now see that v is in the null space of AB.