Math 544, Exam 3, Spring, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in the vector space V are <u>linearly independent</u> if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^p c_i v_i = 0$ are $\overline{c_1 = c_2 = \cdots = c_p} = 0$.

(2) **Let A be an $n \times m$ matrix. Suppose that $v_1, \dots, v_a, w_1 \dots, w_b$ are vectors in $\mathbb{R}^n$ with $v_1, \dots, v_a$ linearly independent elements in the null space A, and $Aw_1, \cdots, Aw_b$ linearly independent elements in $\mathbb{R}^m$. Prove that $v_1, \dots, v_a, w_1, \dots, w_b$ are linearly independent.**

Suppose that $c_1, \dots, c_{a+b}$ are numbers with

$$0 = \sum_{i=1}^{a} c_i v_i + \sum_{j=1}^{b} c_{j+a} w_j. \tag{1}$$

We will demonstrate that every constant $c_1, \dots, c_a, c_{a+1}, \dots, c_{a+b}$ MUST be zero.

Consider the product

$$0 = A(0) = A\left(\sum_{i=1}^{a} c_i v_i + \sum_{j=1}^{b} c_{j+a} w_j\right) = \sum_{i=1}^{a} c_i A v_i + \sum_{j=1}^{b} c_{j+a} A w_j = \sum_{j=1}^{b} c_{j+a} A w_j$$

The last equality holds because the vectors $v_1, \dots, v_a$ are in the null space of A. On the other hand, the hypothesis guarantees that the vectors $Aw_1, \cdots, Aw_b$ are linearly independent. It follows that the constants $c_{a+1}, \dots, c_{a+b}$ all must be zero.

At this point the equation (1) now reads

$$0 = \sum_{i=1}^{a} c_i v_i.$$

The hypothesis guarantees that $v_1, \ldots, v_a$ are linearly independent. We conclude that $c_1, \ldots, c_a$ all must also be zero.

(3) **Let V be the vector space of 4×4 skew-symmetric matrices. Give a basis for V. Recall that the square matrix A is *skew-symmetric* if $A^{\mathrm{T}} = -A$. Justify your answer.**

The matrices

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad M_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

form a basis for V because each of the matrices M_i, with $1 \le i \le 6$, is in V, the matrices $M_1, \ldots, M_6$ are linearly independent, and every element of V is a linear combination of $M_1, \ldots, M_6$.

(4) **Let V be the vector space of polynomials $p(x)$ of degree at most four with the property that $\int_0^1 p(x)dx = 0$. Give a basis for V. Justify your answer.**

The polynomials

$$p_1 = x - \frac{1}{2}, \quad p_2 = x^2 - \frac{1}{3}, \quad p_3 = x^3 - \frac{1}{4}, \quad \text{and} \quad p_4 = x^4 - \frac{1}{5}$$

form a basis for V. It is clear that each p_i is in V. It is also clear that the polynomials $p_1, \ldots, p_4$ are linearly independent. If p is an arbitrary polynomial of degree at most four, then it is clear that p is equal to a linear combination of $p_1, \ldots, p_4$ plus a constant. Observe that the only constant polynomial which is in V is the zero polynomial. It follows that p is in V if and only if p is a linear combination of $p_1, \ldots, p_4$.

(5) **Let**

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 & 3 & 3 \\ 1 & 3 & 4 & 2 & 3 & 3 \\ 1 & 3 & 4 & 1 & 3 & 1 \\ 1 & 3 & 4 & 2 & 3 & 1 \end{bmatrix}.$$

(a) **Find a basis for the null space of A.**

(b) **Find a basis for the column space of A.**

(c) **Find a basis for the row space of A.**

(d) **Express each column of A in terms of your answer to (5b).**

(e) **Express each row of A in terms of your answer to (5c).**

Replace Row 2 with Row 2 minus Row 1,
replace Row 3 with Row 3 minus Row 1, and
replace Row 4 with Row 4 minus Row 1 to obtain

$$\begin{bmatrix} 1 & 3 & 4 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Move Row 2 to the bottom to obtain

$$\begin{bmatrix} 1 & 3 & 4 & 2 & 3 & 3 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiply Row 2 by -1 and
multiply Row 3 by $-1/2$ to obtain

$$\begin{bmatrix} 1 & 3 & 4 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Replace Row 1 with Row 1 minus 2 times Row 2 to obtain

$$\begin{bmatrix} 1 & 3 & 4 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Replace Row 1 with Row 1 plus Row 3 and
replace Row 2 with Row 2 minus 2 times Row 1 to obtain

$$B = \begin{bmatrix} 1 & 3 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is in reduced row echelon form. The matrices A and B have the same null space. It is easy to record a basis for the null space of B. The null space of B is the set of column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

with $Bx = 0$. The variables x_1, x_4, x_6 correspond to leading ones; these are the dependent variables. The variables x_2, x_3, x_5 are free to take any value. We read the equations $Bx = 0$ as

$$\begin{array}{rcccc} x_1 = & -3x_2 & -4x_3 & -3x_5 \\ x_2 = & x_2 & & \\ x_3 = & & x_3 & \\ x_4 = & 0 & & \\ x_5 = & & & x_5 \\ x_6 = & 0 & & \end{array}$$

Thus, the vectors

$$\boxed{v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

form a basis for the null space of A. Indeed, every element in the null space of A is a linear combination of v_1, v_2, and v_3 and v_1, v_2, v_3 are linearly independent.

The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad A_{*,6} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}}$$

form a basis for the column space of A. (I use $A_{*,j}$ to denote column j of A.)

The vectors

$$\boxed{\begin{array}{rl} B_{1,*} = & \begin{bmatrix} 1 & 3 & 4 & 0 & 3 & 0 \end{bmatrix} \\ B_{2,*} = & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ B_{3,*} = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}}$$

are a basis for the row space of A. (I use $B_{i,*}$ to denote row i of B.) Observe that

$$\boxed{\begin{aligned} A_{*,1} &= A_{*,1} \\ A_{*,2} &= 3A_{*,1} \\ A_{*,3} &= 4A_{*,1} \\ A_{*,4} &= A_{*,4} \\ A_{*,5} &= 3A_{*,1} \\ A_{*,6} &= A_{*,6} \end{aligned}}$$

Observe that

$$\boxed{\begin{aligned} A_{1,*} &= 1B_{1,*} + 2B_{2,*} + 3B_{3,*} \\ A_{2,*} &= 1B_{1,*} + 2B_{2,*} + 3B_{3,*} \\ A_{3,*} &= 1B_{1,*} + 1B_{2,*} + 1B_{3,*} \\ A_{4,*} &= 1B_{1,*} + 2B_{2,*} + 1B_{3,*} \end{aligned}}$$