Math 544, Exam 3, Fall, 2020

IF IT IS NECESSARY FOR YOU TO LEAVE THE ROOM DURING THE EXAM, PLEASE LET ME KNOW WHEN YOU LEAVE, WHEN YOU RETURN, AND LEAVE YOUR PHONE WITH ME while you are gone.

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, put your solutions in order, then send a picture of your solutions to

kustin@math.sc.edu

The exam is worth 50 points. Please make your work coherent, complete, and correct.

Recall that for each non-negative integer d, $\mathcal{P}_d$ is the vector space of all polynomials of degree d or less in one variable with real number coefficients.

(1) **Let A be a nonsingular $n\times n$ matrix and B be an $n\times n$ matrix? Answer each question. If the answer is "yes", then prove the statement. If the answer is "no", give an example.**

(a) (10 points) **Does the column space of AB have to equal the column space of B?**

$\boxed{\text{no}}$ If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; and therefore B and AB have different column spaces. (The column space of B is the "x-axis". The column space of AB is the "y-axis".)

(b) (10 points) **Does the null space of AB have to equal the null space of B?**

$\boxed{\text{yes}}$ If v is in the null space of B, then $Bv = 0$; so $ABv = 0$ and v is in the null space of AB. If v is in the null space of AB, then $ABv = 0$; however A is non-singular, so Bv also has to be zero; hence, v is in the null space of B.

(c) (10 points) **Does the rank of AB have to equal the rank of B?**

$\boxed{\text{yes}}$ Part (b) proves that B and AB have the same nullity. These matrices also have the same number of columns. The rank-nullity Theorem now guarantees that B and AB have the same rank.

(2) (10 points) **Let $V = \{p \in \mathcal{P}_5 \mid p(1) = 0,\ p'(1) = 0,$ and $p''(1) = 0\}$. Give a basis for V. Prove your answer.**

We prove that the polynomials

$$\boxed{p_3(x) = (x-1)^3,\ p_4(x) = (x-1)^4,\ p_5(x) = (x-1)^5 \text{ form a basis for } V.}$$

First of all, it is clear that $p_3(x)$, $p_4(x)$, and $p_5(x)$ are all in V.

It is not hard to see that the polynomials $p_3(x)$, $p_4(x)$, and $p_5(x)$ are linearly independent. Indeed, if c_3, c_4, c_5 are numbers with

$$c_3p_3(x) + c_4p_4(x) + c_5p_5(x) \tag{1}$$

equal to the zero polynomial, then the coefficient of x^5 in (1) is zero. But this coefficient is c_5. Thus,

$$c_3p_3(x) + c_4p_4(x) \tag{2}$$

is equal to the zero polynomial. Hence the coefficient of x^4 in (2) is zero. This coefficient is c_4. Thus,

$$c_3p_3(x) \tag{3}$$

is equal to the zero polynomial. Hence the coefficient of x^3 in (3) is zero. This coefficient is c_3. We have shown that c_3, c_4, c_5 all must be zero. We conclude that the polynomials $p_3(x)$, $p_4(x)$, and $p_5(x)$ are linearly independent.

We finish the argument by proving that[1] $\dim V \le 3$. Consider the vector spaces

$$V \subsetneq V_3 \subsetneq V_4 \subsetneq \mathcal{P}_5,$$

where

$$V_3 = \{p \in \mathcal{P}_5 \mid p(1) = 0 \text{ and } p'(1) = 0\},$$

and

$$V_4 = \{p \in \mathcal{P}_5 \mid p(1) = 0\}.$$

We know that $\dim \mathcal{P}_5 = 6$ because $1, x, x^2, x^3, x^4, x^5$ is a basis for $\mathcal{P}_5$.

We know $V_4 \subsetneq \mathcal{P}_5$ because 1 is in $\mathcal{P}_5 \setminus V_4$. Thus $\dim V_4 \le 5$.

We know $V_3 \subsetneq V_4$ because $(x-1)$ is in $V_4 \setminus V_3$. Thus, $\dim V_3 \le 4$.

We know $V \subsetneq V_3$ because $(x-1)^2$ is in $V_3 \setminus V$. Thus, $\dim V_3 \le 3$.

The proof is complete.

(3) (10 points) **Let U and V be finite dimensional subspaces of the vector space W. Recall that $U \cap V$ is the vector space**

$$U \cap V = \{w \in W \mid w \in U \textbf{ and } w \in V\}.$$

Let $w_1, \dots, w_r$ be a basis for $U \cap V$. Let $u_1, \dots, u_s$ be in U and $v_1, \dots, v_t$ in V be vectors so that the vectors $w_1, \dots, w_r, u_1, \dots, u_s$ are linearly

[1] If p_3, p_4, p_5 is a linearly independent set of size three in a vector space of dimension at most three, then the vector space has dimension equal to exactly three and the linearly independent set is in fact a basis.

independent and the vectors $w_1, \ldots, w_r, v_1, \ldots, v_t$ are linearly independent. Prove that the vectors $w_1, \ldots, w_r, u_1, \ldots, u_s, v_1, \ldots, v_t$ are linearly independent.

We prove that $w_1, \ldots, w_r, u_1, \ldots, u_s, v_1, \ldots, v_t$ are linearly independent. Suppose there are numbers a_i, b_j, and c_k such that

$$\sum_{i=1}^{r} a_i w_i + \sum_{j=1}^{s} b_j u_j + \sum_{k=1}^{t} c_k v_k = 0. \tag{4}$$

The sum

$$\sum_{i=1}^{r} a_i w_i + \sum_{j=1}^{s} b_j u_j = -\sum_{k=1}^{t} c_k v_k$$

is in $U \cap V$. The vectors $w_1, \ldots, w_r$ are a basis for $U \cap V$; hence, there are numbers $d_1, \ldots, d_r$ so that

$$-\sum_{k=1}^{t} c_k v_k = \sum_{i=1}^{r} d_i w_i.$$

However, the vectors $w_1, \ldots, w_r, v_1, \ldots, v_t$ are linearly independent; hence, $c_1, \ldots, c_t$ are all zero!

Now equation (4) says that

$$\sum_{i=1}^{r} a_i w_i + \sum_{j=1}^{s} b_j u_j = 0.$$

The vectors $w_1, \ldots, w_r, u_1, \ldots, u_s$ are linearly independent; thus each a_i and each b_j is zero. We conclude that $w_1, \ldots, w_r, u_1, \ldots, u_s, v_1, \ldots, v_t$ are linearly independent.