Math 544, Exam 3, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. **Let** $A = \begin{bmatrix} 1 & 4 & -2 & 1 & 5 & 5 & 5 \\ 1 & 4 & -2 & 2 & 8 & 9 & 7 \\ 2 & 8 & -4 & 3 & 13 & 14 & 0 \\ 3 & 12 & -6 & 5 & 21 & 23 & 7 \end{bmatrix}$.

   Find a basis for the null space of A. Find a basis for the column space of A. Find a basis for the row space of A. Express each column of A in terms of your basis for the column space. Express each row of A in terms of your basis for the row space. Check your answer.

   Apply $R2 \mapsto R2 - R1$, $R3 \mapsto R3 - 2R1$, $R4 \mapsto R4 - 3R1$ to obtain

$$\begin{bmatrix} 1 & 4 & -2 & 1 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 & 4 & -10 \\ 0 & 0 & 0 & 2 & 6 & 8 & -8 \end{bmatrix}.$$

Apply $R1 \mapsto R1 - R2$, $R3 \mapsto R3 - R2$, $R4 \mapsto R4 - 2R2$ to obtain

$$\begin{bmatrix} 1 & 4 & -2 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 \end{bmatrix}.$$

Apply $R3 \mapsto -(1/12)R3$ to obtain

$$\begin{bmatrix} 1 & 4 & -2 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 \end{bmatrix}.$$

Apply $R1 \mapsto R1 - 3R3$, $R2 \mapsto R2 - 2R3$, $R4 \mapsto R4 + 12R3$ to obtain

$$\begin{bmatrix} 1 & 4 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read that the null space of A is equal to the set of vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that

$$\boxed{v_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{is a basis for the null space of } A;}$$

$$\boxed{c_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, c_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, c_3 = \begin{bmatrix} 5 \\ 7 \\ 0 \\ 7 \end{bmatrix} \quad \text{is a basis for the column space of } A;}$$

$$\boxed{\begin{array}{r} w_1 = [1 \quad 4 \quad -2 \quad 0 \quad 2 \quad 1 \quad 0], \\ w_2 = [0 \quad 0 \quad 0 \quad 1 \quad 3 \quad 4 \quad 0], \\ w_3 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \end{array} \quad \text{is a basis for the row space of } A;}$$

column 1 of A is c_1; column 2 of A is $4c_1$; column 3 of A is $-2c_1$; column 4 of A is c_2; column 5 of A is $2c_1+3c_2$; column 6 of A is $1c_1+4c_2$; and column 7 of A is c_3; row 1 of A is $w_1+w_2+5w_3$; row 2 of A is $w_1+2w_2+7w_3$; row 3 of A is $2w_1+3w_2$; and row 4 of A is $3w_1+5w_2+7w_3$.

2. **Define basis. Use complete sentences. Say everything that has to be said and nothing more.**

A linearly independent subset of a vector space V which also spans V is a basis for V.

3. **Define dimension. Use complete sentences. Say everything that has to be said and nothing more.**

The dimension of a vector space is the number of vectors in a basis for the vector space.

4. **Let A be an $n \times m$ matrix and V be a subspace of $\mathbb{R}^m$. Define N and C to be the following vector spaces**

$$N = \{v \in V | Av = 0\} \quad \textbf{and} \quad C = \{Av \mid v \in V\}.$$

Let $u_1, \ldots, u_p$ be vectors in V with $Au_1, \ldots, Au_p$ a basis for C and let $v_1, \ldots, v_q$ be a basis for N. Prove that the vectors $u_1, \ldots, u_p, v_1, \ldots, v_q$ span V. (You will have to write a proof from scratch. We have not proven this particular statement before.)

Let v be an arbitrary element of V. The vector Av is in C; so Av can be written in terms of the basis $Au_1, \ldots, Au_p$ for C. In other words, there are scalars $\alpha_1, \ldots, \alpha_p$ with $Av = \sum_{i=1}^{p} \alpha_i A u_i$. It follows that $v - \sum_{i=1}^{p} \alpha_i u_i$ is in N. Thus, $v - \sum_{i=1}^{p} \alpha_i u_i$ can be written in terms of the basis $v_1, \ldots, v_q$ for V. In other words, there are scalars $\beta_1, \ldots, \beta_q$ with $v - \sum_{i=1}^{p} \alpha_i u_i = \sum_{j=1}^{q} \beta_j v_j$. Thus, $v = \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j$ is in the span of $u_1, \ldots, u_p, v_1, \ldots, v_q$.

5. **Let $U_1 \subseteq U_2 \subseteq U_3 \subseteq \mathbb{R}^4$ be vector spaces. Suppose v_1, v_2, v_3, v_4 is a basis for $\mathbb{R}^4$, $v_1, v_2, v_3 \in U_3$, $v_4 \notin U_3$; $v_1, v_2 \in U_2$, $v_3 \notin U_2$; and $v_1 \in U_1$, $v_2 \notin U_1$. Tell as much as you can about $\dim U_1$, $\dim U_2$, and $\dim U_3$. Prove any statements that you make.**

We know that $\dim U_3 = 3$, $\dim U_2 = 2$ and $\dim U_1 = 1$. Start with U_3. We are told that U_3 contains three linearly independent vectors (hence $3 \le \dim U_3$) and U_3 is a proper subspace of a 4-dimensional vector space (hence $\dim U_3 < 4$). The only posibility left open is that $\dim U_3 = 3$.

We repeat the above argument twice. We are told that U_2 contains two linearly independent vectors and U_2 is a proper subspace of a 3-dimensional vector space; hence, $2 \le \dim U_2 < 3$. The only option left is $\dim U_2 = 2$.

Finally, we are told that U_1 contains linearly independent set of size one and is a proper subspace of a 2-dimensional vector space. Hence $1 \le \dim U_1 < 2$. The only option left is $\dim U_1 = 1$.