Math 544, Exam 3, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **6** problems. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (8 points) **Let A be a 3×4 matrix. Suppose that there is a vector v_0 with the property that every vector v with the property $Av = 0$ is a multiple of v_0. Is it possible to solve $Ax = b$ for all $b \in \mathbb{R}^3$? Explain thoroughly.**

YES. The hypothesis about v_0 says that the nullity of A is 1. So the rank nullity theorem ensures that the rank of A is 3. It follows that the column space of A is a 3-dimensional subspace of $\mathbb{R}^3$. The only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. So the column space of A is $\mathbb{R}^3$. This is just a different way of saying that it is possible to solve $Ax = b$ for all $b \in \mathbb{R}^3$?

2. (8 points) **Suppose that $W \subseteq V$ are vector spaces and that v_1, v_2, v_3, v_4 is a basis for V. Suppose further, that $v_1 \in W$ but $v_2 \notin W$, $v_3 \notin W$, and $v_4 \notin W$. List all of the possible values for W. Explain thoroughly.**

The dimension of W might be 1, or 2, or 3. We are told that W is a non-zero vector space which properly sits in a four dimensional vector space. Thus, $1 \le \dim W \le 3$. We give three examples to illustrate that all three possibilities can occur. Take $V = \mathbb{R}^4$,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Observe that v_1, v_2, v_3, v_4 are linearly independent vectors in V. Take

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \mathbb{R} \right\}, \quad \text{and}$$

$$W_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle| \, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Observe that W_i is a vector space of dimension i for each i, $v_1 \in W_i$ for each i, but v_2, v_3 and v_4 are not in W_i for each i.

3. (8 points) **Let U and V be subspaces of a vector space W and that $z_1, \dots, z_r$, $u_1, \dots, u_s$, and $v_1, \dots, v_t$ are vectors in W. Suppose further that $z_1, \dots, z_r$ is a basis for the intersection $U \cap V$ of U and V; $z_1, \dots, z_r$, $u_1, \dots, u_s$ is a basis for U and $z_1, \dots, z_r$, $v_1, \dots, v_t$ is a basis for V. Prove that the vectors $z_1, \dots, z_r$, $u_1, \dots, u_s$, $v_1, \dots, v_t$ are linearly independent.**

Fix real numbers $a_1, \dots a_r$, $b_1, \dots, b_s$, and $c_1, \dots, c_t$ with

$$(\bigstar) \qquad \sum_{i=1}^{r} a_i z_i + \sum_{i=1}^{s} b_i u_i + \sum_{i=1}^{t} c_i v_i = 0.$$

Observe that the vector

$$\sum_{i=1}^{r} a_i z_i + \sum_{i=1}^{s} b_i u_i = -\sum_{i=1}^{t} c_i v_i$$

is in the intersection $U \cap V$. We know that $z_1, \dots, z_r$ is a basis for the intersection $U \cap V$; so there are real numbers $d_1, \dots d_r$ with

$$-\sum_{i=1}^{t} c_i v_i = \sum_{i=1}^{r} d_i z_i.$$

It follows that

$$0 = \sum_{i=1}^{t} c_i v_i + \sum_{i=1}^{r} d_i z_i.$$

However, $z_1, \dots, z_r$, $u_1, \dots, u_s$ is a basis for U; so $z_1, \dots, z_r$, $u_1, \dots, u_s$ are linearly independent and therefore, each c_i and each d_i MUST be zero. Return to ($\bigstar$). We have

$$\sum_{i=1}^{r} a_i z_i + \sum_{i=1}^{s} b_i u_i = 0.$$

The vectors $z_1, \dots, z_r$, $u_1, \dots, u_s$ are linearly independent; hence, each a_i and each b_i MUST be zero.

4. (8 points) **Find a matrix A with AB equal to the identity matrix. You may do the problem anyway you like; in particular, you are welcome to notice that the columns of B form an orthogonal set,**

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We see that

$$B^{\mathrm{T}}B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

Multiply both sides by

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}$$

to see that

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix} B^{\mathrm{T}}B = I.$$

Take

$$A = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix} B^{\mathrm{T}} = \boxed{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{12} & \frac{-3}{12} & \frac{1}{12} & \frac{1}{12} \end{bmatrix}}.$$

5. (9 points) **Let** $A = \begin{bmatrix} 1 & 4 & 1 & 5 & 2 & 4 & 6 \\ 1 & 4 & 2 & 10 & 3 & 6 & 9 \\ 1 & 4 & 1 & 5 & 3 & 6 & 9 \\ 3 & 12 & 4 & 20 & 8 & 16 & 24 \end{bmatrix}$. **Find a basis for the null space of A. Find a basis for the column space of A. Find a basis for the row space of A. Express each column of A in terms of your basis for the column space. Express each row of A in terms of your basis for the row space. Check your answer.**

Replace $R2 \mapsto R2 - R1$, $R3 \mapsto R3 - R1$, and $R4 \mapsto R4 - 3R1$ to obtain:

$$\begin{bmatrix} 1 & 4 & 1 & 5 & 2 & 4 & 6 \\ 0 & 0 & 1 & 5 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 5 & 2 & 4 & 6 \end{bmatrix}$$

Replace $R1 \mapsto R1 - R2$ and $R4 \mapsto R4 - R2$ to obtain:

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 5 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Replace $R1 \mapsto R1 - R3$, $R2 \mapsto R2 - R3$, and $R4 \mapsto R4 - R3$ to obtain:

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The null space of A is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$ such that

$$\begin{aligned} x_1 &= -4x_2 \\ x_2 &= x_2 \\ x_3 &= \qquad -5x_4 \\ x_4 &= \qquad x_4 \\ x_5 &= \qquad\qquad -2x_6 - 3x_7 \\ x_6 &= \qquad\qquad x_6 \\ x_7 &= \qquad\qquad\qquad\quad x_7. \end{aligned}$$

The vectors

$$\boxed{v_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

The vectors

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 8 \end{bmatrix}}$$

are a basis for the column space of A. We see that

$$\boxed{\begin{array}{l} \text{col 1 of } A = u_1 \\ \text{col 2 of } A = 4u_1 \\ \text{col 3 of } A = u_2 \\ \text{col 4 of } A = 5u_2 \\ \text{col 5 of } A = u_3 \\ \text{col 6 of } A = 2u_3 \\ \text{col 7 of } A = 3u_3 \end{array}}$$

A basis for the row space of A is

$$\boxed{\begin{array}{c} w_1 = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ w_2 = \begin{bmatrix} 0 & 0 & 1 & 5 & 0 & 0 & 0 \end{bmatrix}, \\ w_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} \end{array}}$$

We see that

$$\boxed{\begin{array}{l} \text{row 1 of } A = 1w_1 + 1w_2 + 2w_3 \\ \text{row 2 of } A = 1w_1 + 2w_2 + 3w_3 \\ \text{row 3 of } A = 1w_1 + 1w_2 + 3w_3 \\ \text{row 4 of } A = 3w_1 + 4w_2 + 8w_3 \end{array}}$$

6. (9 points) **Find an orthogonal basis for the null space of** $A = \begin{bmatrix} 1 & 2 & 1 & 3 \end{bmatrix}$**. Check your answer.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $u_1 = v_1$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}.$$

Let $u_2 = 5u_2' = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}$. We verify that u_2 is in the null space of A and $u_2^{\mathrm{T}} u_1 = 0$. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{30} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -5 \\ -10 \\ -5 \\ 10 \end{bmatrix} = \frac{5}{10} \begin{bmatrix} -1 \\ -2 \\ -1 \\ 2 \end{bmatrix}.$$

Let

$$u_3 = 10u_3' = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 2 \end{bmatrix}.$$

Thus

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 2 \end{bmatrix}}$$

is an orthogonal basis for the null space of A. Be sure to verify that $Au_i = 0$ and $u_i^{\mathrm{T}} u_j = 0$ for $i \neq j$.