Math 544, Exam 2, Spring, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Problems 1 and 2 are worth 9 points each. Problems 3 – 6 are worth 8 points each. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^m$ are linearly independent if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^p c_i v_i = 0$ are $c_1 = c_2 = \dots = c_p = 0$.

(2) **Define "nonsingular". Use complete sentences. Include everything that is necessary, but nothing more.**

The $n \times n$ matrix A is nonsingular if the only vector $v \in \mathbb{R}^n$ with $Av = 0$ is $v = 0$.

(3) **Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be a nonsingular $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

The vectors Mv_1, Mv_2, Mv_3 are linearly independent.

Proof. Suppose c_1, c_2, c_3 are numbers with

$$c_1 M v_1 + c_2 M v_2 + c_3 M v_3 = 0.$$

Use the property of scalars and the fact that matrix multiplication distributes over addition to see that

$$M(c_1 v_1 + c_2 v_2 + c_3 v_3) = 0.$$

The matrix M is nonsingular; hence, the only vector w with $Mw = 0$ is $w = 0$. Thus, $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. On the other hand, the vectors v_1, v_2, v_3 are linearly independent. It follows that c_1, c_2, c_3 must all be zero. We have proven that Mv_1, Mv_2, Mv_3 are linearly independent. $\square$

(4) **Let A be a square matrix, v_1 and v_2 be non-zero vectors with $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where λ_1 and λ_2 are real numbers with $\lambda_1 \neq \lambda_2$. Prove that the vectors v_1, v_2 are linearly independent.**

Suppose

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiply both sides of (1) by A to get

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0. \tag{2}$$

Multiply both sides of equation (1) by λ_2 to get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0. \tag{3}$$

Subtract (2) minus (3) to get

$$c_1(\lambda_1 - \lambda_2)v_1 = 0.$$

The vector v_1 is not zero. If a scalar times v_1 is zero, then the scalar must be zero. Thus, the scalar $c_1(\lambda_1 - \lambda_2) = 0$. But, $(\lambda_1 - \lambda_2)$ is not zero; so, c_1 must be zero. Equation (1) now says that $c_2 v_2 = 0$. The vector v_2 is not zero; so, the scalar c_2 must be zero.

(5) Let a and b be fixed vectors in $\mathbb{R}^3$. Consider

$$W = \left\{ x \in \mathbb{R}^3 \middle| a^{\mathrm{T}} x = 0 \quad \text{and} \quad b^{\mathrm{T}} x = 0 \right\}.$$

Is the set W a vector space? Explain thoroughly.

This W is a vector space. Indeed, this W is the null space of

$$\left[\begin{array}{c} a^{\mathrm{T}} \\ \hline b^{\mathrm{T}} \end{array}\right].$$

(6) Solve the system of equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}.$$

If $Ax = b$ has more than one solution, then give the general solution, four particular solutions, and check that your particular solutions work.

We consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 9 & 13 \\ 1 & 2 & 3 & 2 & 13 & 20 \\ 2 & 4 & 6 & 3 & 22 & 33 \end{array}\right]$$

Replace row 2 with row 2 minus row 1.
Replace row 3 with row 3 minus 2 times row 1.
Obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 9 & 13 \\ 0 & 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 4 & 7 \end{array}\right]$$

Replace row 1 with row 1 minus row 2, and
replace row 3 with row 3 minus row 2 to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The most recent augmented matrix corresponds to the equations:

$$\begin{array}{rr} x_1 + 2x_2 + 3x_3 \quad & +5x_5 = 6 \\ x_4 & +4x_5 = 7 \end{array}$$

These equations tell us that x_1 and x_4 are dependent variables and x_2, x_3, and x_5 are free to take any value. The general solution of $Ax = b$ is

$$\boxed{\left\{\begin{array}{lllll} x_1 = & 6 & -2x_2 & -3x_3 & -5x_5 \\ x_2 = & & x_2 & & \\ x_3 = & & & x_3 & \\ x_4 = & 7 & & & -4x_5 \\ x_5 = & & & & x_5, \\ \multicolumn{5}{l}{\text{where } x_2,\ x_3, \text{ and } x_5, \text{ are free to take any value.}} \end{array}\right.}$$

One could also write that the general solution of $Ax = b$ is

$$\boxed{\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \;\middle|\; \begin{array}{l} \text{where } x_2,\ x_3, \\ \text{and } x_5, \\ \text{are free to} \\ \text{take any value.} \end{array} \right\}}$$

When $x_2 = x_3 = x_5 = 0$, then the solution is

$$v_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_2 = 1$ and $x_3 = x_5 = 0$, then the solution is

$$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_3 = 1$ and $x_2 = x_5 = 0$, then the solution is

$$v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_5 = 1$ and $x_2 = x_3 = 0$, then the solution is

$$v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

We check that

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}. \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}. \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}. \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}. \checkmark$$

We have verified that

$$\boxed{v_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}}$$

are particular solutions of $Ax = b$.