Math 544, Exam 2, Fall, 2020

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

kustin@math.sc.edu

You should KEEP this piece of paper. If possible: put the problems in order before you take your picture. (Use as much paper as necessary).

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Define "column space". Use complete sentences. Include everything that is necessary, but nothing more.**

The <u>column space</u> of the $m \times n$ matrix A is

$$\{Av \mid v \in \mathbb{R}^n\}.$$

(2) **Let A and B be $n \times n$ matrices. Is the column space of AB always contained in the column space of A? If yes, prove the assertion. If no, give an example.**

Yes. If w is in the column space of AB, then $w = ABv$ for some vector v. Hence, $w = A(Bv)$ where Bv is a vector. We have shown that w is in the column space of A.

(3) **Let A and B be $n \times n$ matrices. Is the column space of A always contained in the column space of AB? If yes, prove the assertion. If no, give an example.**

No. Let A be the identity matrix and B be the zero matrix. The column space of A is $\mathbb{R}^n$, but the column space of AB consists of the zero vector. The vector space $\mathbb{R}^n$ is not contained in the set which consists of one zero vector.

(4) **Let A and B be $n \times n$ matrices, with B non-singular. Is the column space of A always contained in the column space of AB? If yes, prove the assertion. If no, give an example.**

Yes. Let w be an arbitrary vector in the column space of A. So $w = Av$ for some vector $v \in \mathbb{R}^n$. The matrix B has an inverse (by the Nonsingular Matrix Theorem); consequently, $w = AB(B^{-1}v)$; hence, w is in the column space of AB.

(5) **Let A be an $n \times n$ matrix. Suppose that $v_1, \ldots, v_s$ are vectors in $\mathbb{R}^n$ with $Av_1, \ldots, Av_s$ linearly independent, and that $w_1, \ldots, w_t$ are linearly independent vectors in the null space of A. Prove that $v_1, \ldots, v_s, w_1, \ldots, w_t$ are linearly independent vectors.**

Suppose $a_1, \ldots, a_s, b_1, \ldots, b_t$ are numbers with

$$\sum_{i=1}^{s} a_i v_i + \sum_{j=1}^{t} b_j w_j = 0. \tag{1}$$

Multiply by A, and use the properties of scalars, to see that

$$\sum_{i=1}^{s} a_i A v_i + \sum_{j=1}^{t} b_j A w_j = 0.$$

Of course, $Aw_j = 0$ for all j; hence

$$\sum_{i=1}^{s} a_i A v_i = 0.$$

The vectors $Av_1, \ldots, Av_s$ are linearly independent; thus, $a_i = 0$ for all i. Now equation (1) says that $\sum_{j=1}^{t} b_j w_j = 0$. The vectors $w_1, \ldots, w_t$ are linearly independent; hence $b_j = 0$ for all j.

We have proven that the only numbers $a_1, \ldots, a_s, b_1, \ldots, b_t$ for which (1) holds has $a_i = 0$ and $b_j = 0$ for all i and j. We conclude that the vectors $v_1, \ldots, v_s, w_1, \ldots, w_t$ are linearly independent.